

Holographic Entanglement for Chern-Simons Terms

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ABSTRACT: We derive the holographic entanglement entropy contribution from pure and mixed gravitational Chern-Simons(CS) terms in AdS_{2k+1} . This is done through two different methods : first, by a direct evaluation of CS action in a holographic replica geometry and second by a descent of Dong's derivation applied to the corresponding anomaly polynomial. In lower dimensions ($k = 1, 2$), the formula coincides with the Tachikawa formula for black hole entropy from gravitational CS terms. New extrinsic curvature corrections appear for $k \geq 3$: we give explicit and concise expressions for the two pure gravitational CS terms in AdS_7 and present various consistency checks, including agreements with the black hole entropy formula when evaluated at the bifurcation surface.

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1 Introduction

1.1 Overview

AdS/CFT takes a state of CFT and recasts into geometry on the AdS side. While we understand a lot about how this dictionary works, a clear cut algorithm on the field theory side to construct the dual geometry is missing. Whatever the final algorithm be, it is becoming increasingly clear that entanglement measures in the CFT state should play a crucial role in the final answer [1, 2].

Recent years have seen an expanding interest in entanglement measures in holography. Consequently, we will refrain from giving an exhaustive list and confine ourselves to the works crucial to understanding our paper. We begin with the work by Lewkowycz-Maldacena [3] which clarified the subtleties in holographic replica trick and proved Ryu-Takayanagi formula [4, 5] within their framework.¹ This was followed by Dong [8] (and partly by Camps in [9]) who used the holographic replica trick to derive an interesting formula for the entanglement entropy of higher derivative theories with a Lagrangian density depending only on Riemann tensor.² The holographic entanglement entropy formula due to Dong takes the schematic

¹See also an earlier work [6] toward the proof of Ryu-Takayanagi formula and a criticism of this earlier proof can be found in [7].

²Works preceding Dong dealing with higher derivative actions include [10–17]. For some recent works relevant to the higher-derivative correction to holographic entanglement entropy formula, see [18–30]. See [28] for a generalization to the case involving derivatives of the Riemann tensor.

form

$$S_{\text{ent}} = \int_{\Sigma} \left[\left(\frac{\partial \mathbf{L}}{\partial \mathbf{R}} \right) + \mathbf{K} \left(\frac{\partial^2 \mathbf{L}}{\partial \mathbf{R} \partial \mathbf{R}} \right) \mathbf{K} \right], \quad (1.1)$$

where the \mathbf{L} is the gravity Lagrangian density, \mathbf{R} is the Riemann curvature and \mathbf{K} 's denote the extrinsic curvatures.

The above formula and its descendants end up producing a bewildering series of extrinsic curvature terms which fall out at the end of a somewhat technical computation. Unlike the leading Wald term, which Wald [31] and Iyer-Wald [32] famously interpreted as a Noether charge and can be written in a very simple form, it is unclear how should one physically interpret these extrinsic corrections. It is also pertinent to ask what would be a good way of organizing and rewriting these extrinsic terms which makes their physical content transparent. Ideally, we would like to have a structural understanding of these terms using general properties of (holographic) entanglement entropy and give alternate derivations which clarify why they should be there.³ It would also be useful to construct independent checks which does not rely on holographic replica trick.

In this article, we will focus on a special class of higher derivative terms : the pure and mixed gravitational Chern-Simons (CS) terms. The universal nature of Chern-Simons terms along with their relation to anomalies of the dual CFT make them a perfect laboratory to study the extrinsic corrections mentioned above. Given that the Wald-Tachikawa entropy [33, 34]⁴ of large charged rotating AdS black holes has been reproduced by analyzing CFT anomalies [39, 40], it is plausible that the extrinsic corrections à la Dong [8] can also be explained by a careful study of entanglement entropy in the presence of anomalies. This work was initiated in the hope that if we examine the extrinsic curvature corrections in the case of Chern-Simons terms, it would help us clarify the structure of these corrections and possibly even derive them from CFT considerations.

One way to think about these extrinsic curvature correction terms in the context of Noether method is to think of them as a particular choice of pre-symplectic current that is natural from the entanglement entropy point of view. From this point of view, the question can be rephrased as to why this particular choice is privileged over others. In a previous paper [38], we have pointed out that in the case of Chern-Simons terms Wald's prescription for pre-symplectic current needs to be modified to preserve covariance under diffeomorphisms and gauge transformations. This, apart from resolving conceptual issues about covariance in previous works [33, 34], also reproduces the correct odd parity Cardy type formula for higher

³Recently it has been suggested in [23] that linearized second law can give an alternative derivation of some of the extrinsic corrections.

⁴We note that the original derivation by Wald was applicable only to covariant Lagrangian densities. This derivation was later generalized to 3d and higher dimensional CS terms on the work by Tachikawa[33]. The special case of 3d gravitational CS terms was however already known from other points of view[35–37]. The covariance issues in Tachikawa's original derivation were pointed out in [34] and eventually resolved in [38] where the general BH entropy formula (see Eq. (5.8)) for arbitrary CS terms was written down. We will refer to this formula as Tachikawa formula in the following.

dimensional black holes [40, 41]. Thus, another motivation for studying extrinsic corrections for Chern-Simons terms is to compare these two deformations of the pre-symplectic current. Given the explicit charged rotating black holes with Chern-Simons terms constructed in [41] and the interesting way in which anomalies are encoded in their geometry, it is plausible to hope that detailed studies of entanglement probes and the extrinsic corrections in these geometries would give us more physical insight into both anomalies and entanglement entropy.

In this paper, we will take the first step towards these goals by repeating the holographic replica calculation for a gravitational theory with CS terms and leave a detailed analysis of its structure to future work. In particular, we will postpone the important problem of how to determine the entangling surface from the bulk equations of motion. Our focus here would be to get the entanglement functional which when integrated over an appropriate surface would give us the entanglement entropy. This computation, while a slight generalization of [8], presents its own peculiar subtleties due to the non-covariance of CS terms. The Dong formula thus does not apply directly to our case. This state of affairs can be remedied in two ways - the first is to just repeat the holographic replica method, face the subtleties carefully and get the corresponding formula with Chern-Simons terms. The case of gravitational CS term on AdS_3 has already been studied in [42]⁵ and it was argued there was no extrinsic curvature dependence over and above the Tachikawa formula for BH entropy [33, 34, 38]. We will extend their computation to higher dimensions : mixed gravitational CS term in AdS_5 along with pure/mixed gravitational CS terms in AdS_7 . While the mixed CS terms are a straightforward generalization of AdS_3 case and show no corrections to the Tachikawa formula, the pure gravitational CS terms in AdS_7 show an interesting dependence on extrinsic curvatures.

While this first method is a direct generalization of Dong's original derivation and a simple abstract formula can be written down for any CS term (See Eq. (4.1)), the complicated Christoffel connection dependence in higher dimensional CS terms makes it more and more tedious to evaluate our formula explicitly as we move higher in dimensions.⁶ Keeping this in mind, we also present an indirect but an easier method to arrive at the same answer for any CS term by using the corresponding anomaly polynomial (or the characteristic class). The idea here is to imagine the CS term I_{CS} as descending from a covariant theory living in one higher dimension with an action $dI_{\text{CS}} = \mathcal{P}_{\text{anom}}$. In the first step, we calculate the holographic replica action in one higher dimension. As we verify in our Appendix C, because of covariance, the answer can also be directly obtained from Dong's formula. In the second step, we write the answer as a pure boundary contribution and identify that boundary contribution with the holographic replica action of CS terms. In all cases computed, our answers by this method matches with a direct evaluation of CS action.

⁵Previous discussions of this problem include [43, 44].

⁶Perhaps a clever reader can exploit special properties of CS terms to simplify our formula and its evaluation.

1.2 Summary of Main Results

For reader's convenience, we summarize here the main results obtained in this paper — the derivation of holographic formulae for Chern-Simons contribution to entanglement entropy. Our results for gravitational and mixed Chern-Simons terms in AdS_{d+1} with $d = 2k$ ($k = 1, 2, 3$) are as follows :⁷

$$\text{AdS}_3 \text{ with } \mathcal{P}_{\text{anom}} = c \text{tr}(\mathbf{R}^2) \quad : \quad S_{\text{ent,CS}} = 2(2\pi)(2c) \int_{\Sigma} \mathbf{\Gamma}_N, \quad (1.2)$$

$$\text{AdS}_5 \text{ with } \mathcal{P}_{\text{anom}} = c \mathbf{F} \wedge \text{tr}(\mathbf{R}^2) \quad : \quad S_{\text{ent,CS}} = 2(2\pi)(2c) \int_{\Sigma} \mathbf{F} \wedge \mathbf{\Gamma}_N, \quad (1.3)$$

$$\text{AdS}_7 \text{ with } \mathcal{P}_{\text{anom}} = c \mathbf{F}^2 \wedge \text{tr}(\mathbf{R}^2) \quad : \quad S_{\text{ent,CS}} = 2(2\pi)(2c) \int_{\Sigma} \mathbf{F}^2 \wedge \mathbf{\Gamma}_N, \quad (1.4)$$

$$\begin{aligned} \text{AdS}_7 \text{ with } \mathcal{P}_{\text{anom}} = c \text{tr}(\mathbf{R}^4) \quad : \quad & S_{\text{ent,CS}} \\ & = 2(2\pi)(4c) \int_{\Sigma} \left[\mathbf{\Gamma}_N \wedge \mathbf{R}_N^2 + [-2\mathbf{R}_N + 3(\bar{\mathbf{K}} \cdot \mathbf{K})] \wedge [(\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}}) + (\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K})] \right. \\ & \quad \left. + (\hat{D}\mathbf{K} \cdot \mathbf{r} \cdot \bar{\mathbf{K}}) - (\hat{D}\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{K}) \right], \end{aligned} \quad (1.5)$$

$$\begin{aligned} \text{AdS}_7 \text{ with } \mathcal{P}_{\text{anom}} = c \text{tr}(\mathbf{R}^2) \wedge \text{tr}(\mathbf{R}^2) \quad : \quad & S_{\text{ent,CS}} \\ & = 2^2(2\pi)(4c) \int_{\Sigma} \left[\mathbf{\Gamma}_N \wedge \mathbf{R}_N^2 + \frac{1}{2} \mathbf{\Gamma}_N \wedge \text{tr}(\mathbf{r}^2) \right. \\ & \quad \left. + 2[-\mathbf{R}_N + (\bar{\mathbf{K}} \cdot \mathbf{K})] \wedge [(\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}}) + (\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K})] \right]. \end{aligned} \quad (1.6)$$

In the above expression, Σ denotes the bulk entangling surface with co-dimension two, \mathbf{F} is $U(1)$ field strength, \mathbf{R} is the curvature two-form, \mathbf{K}_i and $\bar{\mathbf{K}}_i$ are extrinsic curvatures of Σ and \mathbf{r} is the intrinsic curvature two-form of Σ . The $\mathbf{\Gamma}_N$ and $\mathbf{R}_N = d\mathbf{\Gamma}_N$ are the normal bundle connection and field strength while \hat{D} is the covariant derivative associated with $\mathbf{\Gamma}_N$ defined such that⁸

$$\hat{D}\mathbf{K}_i \equiv D\mathbf{K}_i - \mathbf{\Gamma}_N \wedge \mathbf{K}_i, \quad \hat{D}\bar{\mathbf{K}}_i \equiv D\bar{\mathbf{K}}_i + \mathbf{\Gamma}_N \wedge \bar{\mathbf{K}}_i, \quad (1.7)$$

where D is the covariant derivative on Σ . The ‘ \cdot ’ in the above equation denotes the contraction of the indices using the intrinsic metric on Σ . The anomaly coefficient c is a numerical constant. For simplicity, in the course of the computation starting from the next section, we

⁷Due to the simplicity of the expression, we write down the anomaly polynomials $d\mathbf{I}_{\text{CS}} = \mathcal{P}_{\text{anom}}$ instead of Chern-Simons terms themselves.

⁸We provide explicit and detailed expressions for these notations and various other definitions in our appendices.

will set the anomaly coefficient c to be unity. It is straightforward to restore the anomaly coefficients by multiplying our results with c .

The first three formulae are exactly the corresponding Wald-Tachikawa formula for black hole entropy in the presence of CS terms. As explained before, the interesting extrinsic curvature dependence appears only for the pure gravitational CS terms in AdS_7 .

1.3 Outline

The outline of this paper is as follows. We start in §2 by introducing the setup we will investigate throughout this paper: Chern-Simons terms, anomaly polynomials and regularized cone geometry. We also provide a brief review of [8] and then explain how the holographic entanglement entropy formulae can be obtained from the anomaly polynomials and Chern-Simons terms. In §3, we summarize our derivation of the formulae from the anomaly polynomials corresponding to 3d and 7d gravitational Chern-Simons terms. In §4, we present an alternative derivation of the same results directly from the Chern-Simons terms. Finally in §5, we explain a straightforward generalization of the holographic entanglement entropy formula to certain class of mixed Chern-Simons terms, consistency check with the black hole entropy formulae and the frame-dependence of the entanglement entropy in the presence of quantum anomalies. The final section §6 of the main text is devoted for the conclusions and discussion.

The appendices of this paper are organized as follows. Appendix A collects some explicit forms of geometric quantities evaluated on the regularized cone geometry, such as connection one-form, curvature two-form as well as their wedge products. Details of the computation in §3 are given in Appendix B while Appendix C summarizes another way to derive the holographic entanglement entropy formulae by directly applying Dong’s formula to anomaly polynomials, without doing the expansion near the regularized conical defect explicitly. Finally, Appendix D summarizes the direct derivation of the holographic entanglement entropy formula from the 7d single-trace gravitational Chern-Simons terms.

Note Added: While we were finishing this paper, [45] appeared and have substantial overlap in the derivation of holographic entanglement entropy formula for 7d single-trace gravitational Chern-Simons term from the anomaly polynomial. Their published results are consistent with ours. We also note that [46, 47] have some overlap with our paper.⁹

2 Setup and Outline of Computation

In this section, we begin by summarizing the setups we use throughout this paper. We start with a brief review of Chern-Simons terms and their corresponding anomaly polynomials. This is followed by a discussion of the regularized cone geometry on which we will evaluate the anomaly polynomials and Chern-Simons terms to obtain the holographic entanglement entropy formula.¹⁰ Following [8], we then will focus on two types of contributions (the Wald

⁹We would like to thank N. Iqbal and A. C. Wall for correspondence.

¹⁰See Appendix A for a list of geometric quantities evaluated on this background.

term and the extrinsic correction term) to the holographic entanglement entropy formula with a particular focus on our case of interest. We will conclude by summarizing our computational strategy.

2.1 Chern-Simons Term and Anomaly Polynomial

A Chern-Simons term \mathbf{I}_{CS} is an odd dimensional form made by wedging together Christoffel connection 1-forms $\mathbf{\Gamma}^\mu{}_\nu$, vector potential 1-forms \mathbf{A} and their exterior derivatives $d\mathbf{\Gamma}^\mu{}_\nu$ and $d\mathbf{A}$. The combination is chosen such that under gauge transformation and diffeomorphism, it transforms by an exact term. The most straightforward way to characterize the Chern-Simons term is by the corresponding anomaly polynomial defined by $\mathcal{P}_{\text{anom}} = d\mathbf{I}_{\text{CS}}$. This anomaly polynomial is by far simpler and is written only in terms of covariant quantities (i.e. field strength two-form and Riemann curvature two-form).

The Chern-Simons term corresponding to any given anomaly polynomial can be written down in a compact way by using the transgression formula (see for example [34]). We will begin our discussion with pure gravitational Chern-Simons terms whose anomaly polynomial depends only on the Riemann curvature two-form. We will later generalize to mixed Chern-Simons terms in §5.

Here are the explicit forms of the gravitational Chern-Simons terms and the corresponding anomaly polynomials we will need in §3 and §4. In three-dimensional case, the gravitational Chern-Simons term and the corresponding anomaly polynomial are written as

$$\mathbf{I}_{\text{CS}}^{3d} = \text{tr} \left(\mathbf{\Gamma} \wedge \mathbf{R} - \frac{1}{3} \mathbf{\Gamma}^3 \right), \quad \mathcal{P}_{\text{anom}}^{3d} = \text{tr}(\mathbf{R}^2), \quad (2.1)$$

while for $7d$ case there are two types of gravitational Chern-Simons terms: the first one is in the form of the single-trace

$$\mathbf{I}_{\text{CS}}^{7d, \text{single}} = 4 \int_0^1 dt \text{tr} [\mathbf{\Gamma} \wedge (t\mathbf{R} + t(t-1)\mathbf{\Gamma}^2)^3], \quad (2.2)$$

$$\mathcal{P}_{\text{anom}}^{7d, \text{single}} = \text{tr}(\mathbf{R}^4), \quad (2.3)$$

while the second one is double-trace

$$\mathbf{I}_{\text{CS}}^{7d, \text{double}} = \text{tr} \left(\mathbf{\Gamma} \wedge \mathbf{R} - \frac{1}{3} \mathbf{\Gamma}^3 \right) \wedge \text{tr}(\mathbf{R}^2), \quad \mathcal{P}_{\text{anom}}^{7d, \text{double}} = \text{tr}(\mathbf{R}^2) \wedge \text{tr}(\mathbf{R}^2). \quad (2.4)$$

Here $\mathbf{\Gamma}^\mu{}_\nu$ is the connection one-form and $\mathbf{R}^\mu{}_\nu = d\mathbf{\Gamma}^\mu{}_\nu + \mathbf{\Gamma}^\mu{}_\rho \wedge \mathbf{\Gamma}^\rho{}_\nu$ is the curvature two-form. The trace in the above expression is taken for the matrix indices carried by the wedge products of these quantities. For example, $\text{tr}(\mathbf{R}^2) = \mathbf{R}^\mu{}_\nu \wedge \mathbf{R}^\nu{}_\mu$.

2.2 Deriving Holographic Entanglement Entropy Formulae: A Review

Our goal in this paper is to derive holographic entanglement entropy formulae for Chern-Simons terms by using the argument of Lewkowycz-Maldacena [3]. The main idea of [3] is the following : start with the geometry near the regularized conical geometry with conical

deficit $2\pi(n-1)$, expand the action on this background around $n=1$ (equivalently around $\epsilon=0$ where $\epsilon \equiv 1-1/n$) and then pick up the linear terms in $n-1$ (equivalently, linear term in ϵ), which corresponds to the holographic entanglement entropy functional. In [8] (see also [9]), this argument is generalized to any general higher-derivative theory whose Lagrangian is a polynomial of Riemann tensor only.

Since the argument has been very well discussed in [3] and reviewed in [8, 42], we will just be brief here and refer the interested readers to these papers for more details. The essential idea is that in the dual CFT, one starts by computing the Rényi entropy for any positive integer n and then take the $n \rightarrow 1$ limit to obtain the entanglement entropy:

$$S_{\text{ent}} = \lim_{n \rightarrow 1} \frac{1}{1-n} [\log Z_n - n \log Z_1] , \quad (2.5)$$

where the partition function on a replica manifold M_n is given by Z_n . In the AdS/CFT context, Z_n is interpreted as the bulk gravitational partition function for a geometry B_n that asymptotes to M_n , that is, $Z_n = \exp(I_{\text{grav}}[B_n])$ where $I_{\text{grav}}[B_n]$ is the on-shell gravitational action on B_n in the semiclassical limit. One convenient method introduced in [3] is to extract S_{ent} using a regularized cone with a conical deficit $2\pi(n-1)$ along a co-dimension two surface in the bulk:¹¹

$$S_{\text{ent}} = \partial_\epsilon \left(I_{\text{cone}}^{\text{inside}} \right) \Big|_{\epsilon=0} , \quad (2.6)$$

where $I_{\text{cone}}^{\text{inside}}$ is the action evaluated near the tip of the regularized cone.

The problem now is to evaluate the right-hand-side of this equation. Below, after introducing the explicit metric of the regularized cone geometry following the notation of [8], we will briefly review the argument in [8] and then explain the origin of non-trivial contributions to holographic entanglement entropy. We also argue that, for the setups in this paper, we can simplify the evaluation thanks to the simple wedge-product structure of Chern-Simons terms and anomaly polynomials.

2.3 Metric of Regularized Cone Geometry

Following the notation of [8], we write down the metric near the tip of the regularized cone geometry in $(D+2)$ -dimensional Euclidean spacetime up to the second order of (z, \bar{z}) -expansion as

$$\begin{aligned} ds^2 = & e^{2A} (dzd\bar{z} + e^{2A} T (\bar{z}dz - zd\bar{z})^2) + \left(g_{ij} + 2K_{aij} x^a + Q_{abij} x^a x^b \right) dy^i dy^j \\ & + 2ie^{2A} (U_i + V_{ai} x^a) (\bar{z}dz - zd\bar{z}) dy^i + \dots , \end{aligned} \quad (2.7)$$

¹¹We note that our notation for the Euclidean action deviates from [8] by a minus sign.

or, in terms of components,¹²

$$\begin{aligned} G_{zz} &= e^{4A} T \bar{z}^2, & G_{\bar{z}\bar{z}} &= e^{4A} T z^2, & G_{z\bar{z}} &= \frac{1}{2} e^{2A} - e^{4A} T |z|^2, \\ G_{iz} &= i e^{2A} (U_i + V_{ai} x^a) \bar{z}, & G_{i\bar{z}} &= -i e^{2A} (U_i + V_{ai} x^a) z, \\ G_{ij} &= g_{ij} + 2K_{aij} x^a + Q_{abij} x^a x^b. \end{aligned} \quad (2.8)$$

Here $(x^1, x^2) = (z, \bar{z})$ are the coordinates transverse to the co-dimension two conical defect, while y^i ($i = 1, 2, \dots, D$) are the ones along it. When we evaluate some quantities like the curvature two-form on this metric, we will simply say that we evaluate them on the regularized cone geometry. In the evaluation of the anomaly polynomials on this geometry, as we will explain shortly and work out in §3, D is taken to be d (where d is a positive even integer) and y^d is set to the coordinate of a half line (i.e. $y^d \in I = [0, \infty)$). When we evaluate the Chern-Simons terms directly as in §4, on the other hand, D is set to $(d - 1)$.¹³ In the above metric, the functions $T, g_{ij}, K_{aij}, Q_{abij}, U_i, V_{ai}$ depend only on y^i , not on z and \bar{z} . The regularization function A is given by

$$A = -\frac{\epsilon}{2} \log(|z|^2 + a_{reg}^2), \quad (2.9)$$

where $\epsilon = 1 - 1/n$ and a_{reg} is a regularization parameter taken to $a_{reg} \rightarrow 0$ at the end of the computation.

The inverse metric up to the first order in the (z, \bar{z}) -expansion is given by

$$\begin{aligned} G^{zz} &= G^{\bar{z}\bar{z}} = 0, & G^{z\bar{z}} &= 2e^{-2A}, \\ G^{iz} &= 2i g^{ik} U_k z, & G^{i\bar{z}} &= -2i g^{ik} U_k \bar{z}, \\ G^{ij} &= g^{ij} - 2g^{ik} g^{jl} K_{akl} x^a. \end{aligned} \quad (2.10)$$

Here g^{ij} is the inverse of g_{ij} (i.e. $g_{ij} g^{jk} = \delta_i^k$). As we will see, the inverse metric up to this order is enough for our purpose.

It is also useful to define the normal-bundle connection $(\Gamma_N)_i$ and its field strength as

$$(\Gamma_N)_j \equiv (-2iU_j), \quad (R_N)_{jk} \equiv 2\partial_{[j}(\Gamma_N)_{k]}. \quad (2.11)$$

We also define the differential forms for them as $\mathbf{\Gamma}_N \equiv (\Gamma_N)_i dx^i$ and $\mathbf{R}_N \equiv d\mathbf{\Gamma}_N$. See Appendix C.6 for more details on their standard definitions.

¹²In a general higher derivative gravity theory, there exists an ambiguity in the definitions of T, Q and V . This has been termed the “splitting problem” by [29, 48]. It is expected that equations of motion will resolve such an ambiguity. In our case, as we will show in the bulk of the paper, T, Q and V will not contribute to the entanglement entropy and hence the splitting problem does not arise in our setup.

¹³Throughout this paper, spacetime indices are denoted with Greek letters μ, ν, ρ, \dots . For simplicity, we use this notation for both before and after we uplift the spacetime by one-dimension to deal with the anomaly polynomials.

2.3.1 Review of [8]

To derive the holographic entanglement entropy formula, the central question is how one can practically collect the terms linear in ϵ when the action on the regularized cone background is evaluated. When the Lagrangian is a polynomial of Riemann tensor only, generalization of the argument by [3] is carried out in [8]. The point of [8] is that there are two types of sources for the order- ϵ contribution:

1. $\partial\bar{\partial}A$ Term

On the regularized cone background, the Riemann tensor contains terms proportional to $\partial\bar{\partial}A$ (here ∂ and $\bar{\partial}$ respectively are derivatives with respect to z and \bar{z}). Since $\partial\bar{\partial}A = -\pi\epsilon\delta^2(z, \bar{z})$, we have the order ϵ terms of the form

$$\int d^2z d^Dy \left[\sqrt{+G} \left(\frac{\partial L}{\partial R_{\mu\nu\rho\sigma}} \right) \right]_{\epsilon=0} (\partial\bar{\partial}A \text{ terms in } R_{\mu\nu\rho\sigma}).$$

2. $\partial A \bar{\partial} A$ Term

Since $\partial A = -\epsilon/(2z)$ and $\bar{\partial} A = -\epsilon/(2\bar{z})$, the product $\partial A \bar{\partial} A$ is of order ϵ^2 . However, the integration with respect to (z, \bar{z}) can compensate to generate order- ϵ terms. To see this, let us define the polar coordinate (ρ, τ) for the 2d space transverse to the conical defect such that $z = \rho e^{i\tau}$. Then, due to the integration (here C is a positive constant)

$$\int d^2z \partial A \bar{\partial} A e^{-CA} = \pi\epsilon^2 \int d\rho \rho \frac{\rho^2}{(\rho^2 + a_{reg}^2)^{2-(\epsilon/2)C}} = -\frac{\pi\epsilon}{C} + \dots, \quad (2.12)$$

the following type of order ϵ terms can show up:

$$2 \int d^2z d^Dy \left[\sqrt{+G} \left(\frac{\partial^2 L}{\partial R_{\mu\nu\rho\sigma} \partial R_{\alpha\beta\gamma\delta}} \right) \right]_{\epsilon=0} (\partial A \text{ terms in } R_{\mu\nu\rho\sigma}) \times (\bar{\partial} A \text{ terms in } R_{\alpha\beta\gamma\delta}).$$

To compute these two contributions, what is needed to keep in the Riemann tensor is:

- Order- ϵ terms of the form ∂A , $\bar{\partial} A$ or $\partial\bar{\partial}A$ with zeroth order terms in (z, \bar{z}) -expansion multiplied,
- Order- ϵ^0 terms which are at the same time the zeroth order in (z, \bar{z}) -expansion.

2.4 Holographic Entanglement Entropy from Anomaly Polynomial: Strategy

Due to its simplicity, we first describe the indirect derivation using anomaly polynomials $\mathcal{P}_{\text{anom}} = d\mathbf{I}_{\text{CS}}$, instead of the direct evaluation of Chern-Simons terms \mathbf{I}_{CS} (which will be described in the next subsection). More concretely, we first uplift by one-dimension and consider the regularized cone geometry in $(d+2)$ -dimensional (Euclidean) spacetime (i.e. y^d in Eq. (2.7) is the coordinate of a half line $I = [0, \infty)$, corresponding to the uplifted direction). We then take the higher dimensional Lagrangian to be the anomaly polynomial $\mathcal{P}_{\text{anom}}$ and evaluate the action on the $(d+2)$ -dimensional regularized cone geometry. After carrying out

ϵ -expansion and collecting the linear terms, we will explicitly see the action to be an exact term. We can then write

$$\int_{\text{Cone}_{\Sigma \times I, n}} \mathcal{P}_{\text{anom}} = \epsilon \int_{\Sigma \times I} d(\mathcal{S}_{\text{ent}}^{CS}) + \dots = \epsilon \int_{\Sigma} \mathcal{S}_{\text{ent}}^{CS}|_{y^d=0} + \dots, \quad (2.13)$$

where we will find compute $\mathcal{S}_{\text{ent}}^{CS}$ explicitly later in the text. The holographic entanglement entropy functional for Cher-Simons terms can then be obtained through $S_{\text{ent,CS}} = \int_{\Sigma} \mathcal{S}_{\text{ent}}^{CS}|_{y^d=0}$.¹⁴ Here, Σ represents the co-dimension two holographic entangling surface in the $(d+1)$ -dimensional bulk where the Chern-Simons term is defined, while $\text{Cone}_{\Sigma \times I, n}$ denotes the region near the tip of the regularized cone geometry (2.7) corresponding to the uplifted conical defect $\Sigma \times I$.

As we have explained the prescription to obtain the holographic entanglement entropy formula from the anomaly polynomials, we next argue how to collect the order- ϵ terms in the (integral of) anomaly polynomials. Since the anomaly polynomials are written with the curvature two-form only (made of Riemann tensor), as in the case of [8], there are two types of contributions at order ϵ . That is, when we evaluate the anomaly polynomials on the regularized cone background, what we need to keep are the terms of the form

- $\partial\bar{\partial}A$ times zeroth order terms in (z, \bar{z}) -expansion,
- $\partial A\bar{\partial}A$ times zeroth order terms in (z, \bar{z}) -expansion.

To evaluate these two types of terms, one needs to explicitly compute the curvature two-form \mathbf{R} on the regularized cone background. Related to this, here we provide one remark which reduces some intermediate computations. First of all, from Eqs. (2.7) and (2.10), we can compute the Christoffel connection up to order- ϵ in the ϵ -expansion and up to first order in the (z, \bar{z}) -expansion. The explicit result is summarized in Eq. (A.1). From this expression, we can easily confirm the following statement for the connection one-form $\mathbf{\Gamma}$: Let us consider the terms of the form ∂A or $\bar{\partial}A$ times zeroth order terms in (z, \bar{z}) -expansion. Such terms appear only in $\mathbf{\Gamma}^z_z$ and $\mathbf{\Gamma}^{\bar{z}}_{\bar{z}}$ as $2e^{-0 \times A}(\partial A dz)$ and $2e^{-0 \times A}(\bar{\partial} A d\bar{z})$, respectively. Therefore, in the curvature two-form, the following is true of leading order terms in (z, \bar{z}) -expansion: (1) there is no term of the form $\partial A\bar{\partial}A$ times zeroth order terms in (z, \bar{z}) -expansion, (2) as for the terms of the form ∂A , $\bar{\partial}A$ or $\partial\bar{\partial}A$, they always occur as $\partial A dz$, $\bar{\partial} A d\bar{z}$ or $\partial\bar{\partial}A dz \wedge d\bar{z}$, times zeroth order terms in (z, \bar{z}) -expansion. To avoid confusion, reader should note that here and elsewhere we count $dz, d\bar{z}$ as zeroth order objects in (z, \bar{z}) -expansion. Because of this simple profile, we can drop the other terms proportional to dz and/or $d\bar{z}$. In particular, we note that Q and V terms of [8] do not contribute to our final answer.

To summarize, starting from Eq. (A.1) what we need to keep in the evaluation of the connection one-form and curvature two-form is:

¹⁴We note that, when we uplifted to $(d+2)$ -dimensions, we extended \mathbf{R}^μ_ν etc. in the following way: the components with at least one of the subscript or superscript set to y^d vanish at $y^d = 0$. We also assumed that \mathbf{R}^μ_ν vanishes at $y^d \rightarrow \infty$ sufficiently fast.

1. For Γ , keep terms of the form
 - Order- ϵ^0 and zeroth order terms in (z, \bar{z}) -expansion,
 - $\partial A dz$ or $\bar{\partial} A d\bar{z}$ times zeroth order terms in (z, \bar{z}) -expansion.
2. For $d\Gamma$, keep terms of the form
 - Order- ϵ^0 and zeroth order terms in (z, \bar{z}) -expansion,
 - $\partial A dz$, $\bar{\partial} A d\bar{z}$ or $\partial \bar{\partial} A dz \wedge d\bar{z}$ times zeroth order terms in (z, \bar{z}) -expansion.
3. For R , keep terms of the form
 - Order- ϵ^0 and zeroth order terms in (z, \bar{z}) -expansion,
 - $\partial A dz$, $\bar{\partial} A d\bar{z}$ or $\partial \bar{\partial} A dz \wedge d\bar{z}$ times zeroth order terms in (z, \bar{z}) -expansion.

Following this, we have computed R etc. The result is summarized in Appendix A.

Here we have explained the strategy to derive the holographic formulae for the entanglement entropy by evaluating the anomaly polynomial on the regularized cone background. This is explicitly worked out in §3 and Appendix B. In Appendix C, on the other hand, we have also explained an alternative way to derive the holographic formulae for entanglement entropy from anomaly polynomials. There, instead of doing the explicit expansion of the anomaly polynomial on the regularized cone geometry, we apply the Dong's entanglement entropy formula in [8] to the anomaly polynomial (since the anomaly polynomial is covariant, Dong's formula is applicable) and reproduce all the results in §3.

2.5 Holographic Entanglement Entropy from Chern-Simons Term: Strategy

Although tedious computation is in general involved at the practical level, in principle, we can also derive the holographic entanglement entropy formulae by evaluating the Chern-Simons terms on the regularized cone background. This computation is worked out in §4 for some examples.

Since the Chern-Simons terms depend on the curvature two-form R as well as the connection one-form Γ , the Dong's formula is not applicable, and one needs to take into account some singularities coming from the connection one-form. Here, we will briefly explain and list the nontrivial contributions to the holographic entanglement entropy when the Chern-Simons terms are directly evaluated on the regularized conical cone geometry. This can be seen as a generalization of [42] to higher dimensional Chern-Simons terms.

As in the case of the anomaly-polynomial-based computation, we have first of all the following two types of sources for order- ϵ term:

- $\partial \bar{\partial} A$ times zeroth order terms in (z, \bar{z}) -expansion,
- $\partial A \bar{\partial} A$ times zeroth order terms in (z, \bar{z}) -expansion.

Since the connection one-form $\mathbf{\Gamma}$ contains one derivative of A only in the form of $\partial A dz$ and $\bar{\partial} A d\bar{z}$, the singularities in $\mathbf{\Gamma}$ do not contribute to the former, but will contribute to the latter.

On top of these, as carefully treated in [42] (see also [35]) for 3d gravitational Chern-Simons term, another type of source for the order- ϵ term exists for the Chern-Simons-term-based computation. Detailed explanation will be given in §4 through concrete examples. Below, we will provide a brief explanation of this type of subtle terms.

To see this new type of contribution, one needs to treat the (z, \bar{z}) -expansion and regularization carefully. Let us consider the following terms that in general appear in the Chern-Simons term evaluated on the regularized cone background:

$$\begin{aligned} \int \partial A dz \wedge d\mathbf{\Gamma} \wedge (\dots) &= \int \partial A dz \wedge d\bar{z} \bar{\partial} \mathbf{\Gamma} \wedge (\dots) + (\dots) \\ &= - \int \partial \bar{\partial} A dz \wedge d\bar{z} \mathbf{\Gamma} \wedge (\dots) + (\dots) \\ &= \pi \epsilon [\mathbf{\Gamma} \wedge (\dots)]|_{0th \text{ in } (z, \bar{z})\text{-expansion}}. \end{aligned} \quad (2.14)$$

(and similarly for $\bar{\partial} A d\bar{z}$). Here we note the $d\mathbf{\Gamma}$ in the above expression is the one before (z, \bar{z}) -expansion is carried out. In the final line we have used the fact that $\partial \bar{\partial} A = -\pi \epsilon \delta^2(z, \bar{z})$. Therefore, as a result of the integration-by-part, this type of terms can show up nontrivially and we need to take them into account as well.

Here is one remark. Even when this new type of contribution is taken into consideration, we can still neglect a lot of terms proportional to dz and/or $d\bar{z}$ in \mathbf{R} and $\mathbf{\Gamma}$ in the same way as the evaluation based on anomaly polynomials. This can be easily seen from the fact that ∂A (resp. $\bar{\partial} A$) always show up in \mathbf{R} and $\mathbf{\Gamma}$ in the form of $\partial A dz$ (resp. $\bar{\partial} A d\bar{z}$), and $d\bar{z}$ (resp. dz) comes from the derivative $d = d\bar{z} \bar{\partial} + dz \partial + \dots$ that we move by integration-by-part (see above). This means that we can still neglect a lot of terms proportional to dz and/or $d\bar{z}$ in the same way as the evaluation based on anomaly polynomials. Therefore, even when we work out with the Chern-Simons terms, the terms in \mathbf{R} etc. summarized in Appendix A are sufficient.

3 Holographic Entanglement Entropy from Anomaly Polynomial

Equipped with the expression of the building blocks $\mathbf{\Gamma}$, \mathbf{R} and their wedge products summarized in Appendix A, following the strategy given in §2.4 now we explicitly evaluate the anomaly polynomials on the regularized cone geometry and extract the terms linear in ϵ . We first start with the anomaly polynomial $\mathcal{P}_{\text{anom}}^{3d}$ for 3d gravitational Chern-Simons term and reproduce the result obtained in [42]. We then proceed to 7d case in which we can consider two types of the gravitational Chern-Simons terms and thus two types of anomaly polynomials. We shall be terse and give only the essential intermediate results in this section. The details of the computations are provided in Appendix B.

3.1 $\text{tr}(\mathbf{R}^2)$

Let us first start with the anomaly polynomial $\mathcal{P}_{\text{anom}}^{3d} = \text{tr}(\mathbf{R}^2)$ corresponding to 3d gravitational Chern-Simons term and evaluate this polynomial on the regularized cone geometry. As explained in §§2.4, there are two types of potential contributions to the holographic entanglement entropy: the terms proportional to $\partial A \bar{\partial} A$ and those proportional to $\partial \bar{\partial} A$. We will first evaluate them separately and combine to get the holographic entanglement entropy formula from the anomaly polynomial. Below are the results:

(1) $\partial \bar{\partial} A$ Term

$$\int_{\text{Cone}_{\Sigma \times I, n}} \mathcal{P}_{\text{anom}}^{3d} |_{\partial \bar{\partial} A} = 8\pi \epsilon \int_{\Sigma \times I} [\mathbf{R}_N - 2(\bar{\mathbf{K}} \cdot \mathbf{K})] . \quad (3.1)$$

(2) $\partial A \bar{\partial} A$ Term

$$\int_{\text{Cone}_{\Sigma \times I, n}} \mathcal{P}_{\text{anom}}^{3d} |_{\partial A \bar{\partial} A} = 16\pi \epsilon \int (\bar{\mathbf{K}} \cdot \mathbf{K}) . \quad (3.2)$$

By summing the above two contributions, we obtain the order- ϵ terms of $\mathcal{P}_{\text{anom}}^{3d}$ on the regularized cone geometry as

$$8\pi \epsilon \int_{\Sigma \times I} \mathbf{R}_N = 8\pi \epsilon \int_{\Sigma} \mathbf{\Gamma}_N . \quad (3.3)$$

Therefore, the holographic entanglement entropy formula for 3d gravitational Chern-Simons term is computed from the anomaly polynomial as

$$S_{\text{ent,CS}}^{3d} = 8\pi \int_{\Sigma} \mathbf{\Gamma}_N , \quad (3.4)$$

which reproduces the result obtained in [42].

3.2 $\text{tr}(\mathbf{R}^4)$

As a next example, we proceed to 7d and consider the single-trace type of gravitational Chern-Simons term $\mathbf{I}_{\text{CS}}^{7d, \text{single}}$. We will derive the holographic entanglement entropy formula from the corresponding anomaly polynomial $\mathcal{P}_{\text{anom}}^{7d, \text{single}} = \text{tr}(\mathbf{R}^4)$. In the same ways as in 3d case, we first evaluate the terms proportional to $\partial \bar{\partial} A$ and $\partial A \bar{\partial} A$ separately and then combine to get the holographic entanglement entropy formula:

(1) $\partial \bar{\partial} A$ Term

By collecting the singular terms proportional to $\partial A \bar{\partial} A dz \wedge d\bar{z}$ from \mathbf{R}^z_z and $\mathbf{R}^{\bar{z}}_{\bar{z}}$, we obtain

$$\begin{aligned}
& \int_{\text{Cone}_{\Sigma \rtimes I, n}} \mathcal{P}_{\text{anom}}^{7d, \text{single}}|_{\partial \bar{\partial} A} \\
&= 16\pi\epsilon \int_{\Sigma \rtimes I} \left[\mathbf{R}_N^3 - 6\mathbf{R}_N^2 \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) - 4\mathbf{R}_N \wedge (\hat{D}\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) + 12\mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 \right. \\
&\quad + 4(\hat{D}\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\hat{D}\mathbf{K} \cdot \bar{\mathbf{K}}) + 4(\hat{D}\bar{\mathbf{K}} \cdot \bar{\mathbf{K}}) \wedge (\hat{D}\mathbf{K} \cdot \mathbf{K}) - 2(\hat{D}\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \hat{D}\mathbf{K}) \\
&\quad \left. + 8(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\hat{D}\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) - 8(\bar{\mathbf{K}} \cdot \mathbf{K})^3 \right]. \tag{3.5}
\end{aligned}$$

(2) $\partial A \bar{\partial} A$ Term

By combining some terms proportional to $\partial A \bar{\partial} A dz \wedge d\bar{z}$ as worked out in detail in Appendix B, we have

$$\begin{aligned}
& \int_{\text{Cone}_{\Sigma \rtimes I, n}} \mathcal{P}_{\text{anom}}^{7d, \text{single}}|_{\partial A \bar{\partial} A} \\
&= 16\pi\epsilon \int \left[2\mathbf{R}_N^2 \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) + \mathbf{R}_N \wedge [-6(\bar{\mathbf{K}} \cdot \mathbf{K})^2 + 2(\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{K})] \right. \\
&\quad + 2(\hat{D}\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\hat{D}\mathbf{K} \cdot \bar{\mathbf{K}}) - 4(\hat{D}\bar{\mathbf{K}} \cdot \bar{\mathbf{K}}) \wedge (\hat{D}\mathbf{K} \cdot \mathbf{K}) + 2(\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{r} \cdot \mathbf{K}) \\
&\quad \left. - 2(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\hat{D}\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) - 6(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{K}) + 8(\bar{\mathbf{K}} \cdot \mathbf{K})^3 \right]. \tag{3.6}
\end{aligned}$$

Summing the contributions above, we obtain

$$\begin{aligned}
& 16\pi\epsilon \int_{\Sigma \rtimes I} d \left[\mathbf{\Gamma}_N \wedge \mathbf{R}_N^2 + [-2\mathbf{R}_N + 3(\bar{\mathbf{K}} \cdot \mathbf{K})] \wedge [(\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}}) + (\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K})] \right. \\
&\quad \left. + (\hat{D}\mathbf{K} \cdot \mathbf{r} \cdot \bar{\mathbf{K}}) - (\hat{D}\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{K}) \right].
\end{aligned}$$

Therefore, up to total-derivative terms, the holographic entanglement entropy formula for the 7d single-trace gravitational Chern-Simons term is obtained from the anomaly polynomial as

$$\begin{aligned}
& S_{\text{ent, CS}}^{7d, \text{single}} \\
&= 2(2\pi)(4) \int_{\Sigma} \left[\mathbf{\Gamma}_N \wedge \mathbf{R}_N^2 + [-2\mathbf{R}_N + 3(\bar{\mathbf{K}} \cdot \mathbf{K})] \wedge [(\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}}) + (\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K})] \right. \\
&\quad \left. + (\hat{D}\mathbf{K} \cdot \mathbf{r} \cdot \bar{\mathbf{K}}) - (\hat{D}\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{K}) \right]. \tag{3.7}
\end{aligned}$$

3.3 $\text{tr}(\mathbf{R}^2) \wedge \text{tr}(\mathbf{R}^2)$

As a final example, here we will derive the holographic entanglement entropy formula for 7d double-trace gravitational Chern-Simons terms by evaluating the anomaly polynomial $\mathcal{P}_{\text{anom}}^{7d, \text{double}} = \text{tr}(\mathbf{R}^2) \wedge \text{tr}(\mathbf{R}^2)$ on the regularized cone background.

(1) $\partial\bar{\partial}A$ Term

The terms of the form $\partial\bar{\partial}Adz \wedge d\bar{z}$ in $\text{tr}(\mathbf{R}^2) \wedge \text{tr}(\mathbf{R}^2)$ can be collected easily since this contribution is essentially the 3d result wedged by the 3d anomaly polynomial with $^{(0)}$ terms in Eq. (A.9) substituted. Because of this “factorization”, using the expression of $\mathcal{P}_{\text{anom}}^{3d}|_{0th}$ given in Eq. (A.37), the result can be obtained easily:

$$\begin{aligned} & \int_{\text{Cone}_{\Sigma \rtimes I, n}} \mathcal{P}_{\text{anom}}^{7d, \text{double}}|_{\partial\bar{\partial}A} \\ &= 16\pi \epsilon \left[\mathbf{R}_N - 2(\bar{\mathbf{K}} \cdot \mathbf{K}) \right] \wedge \left[2\mathbf{R}_N^2 - 8\mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) - 8(\hat{D}\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) + \text{tr}(\mathbf{r}^2) + 8(\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{K}) \right]. \end{aligned} \quad (3.8)$$

(2) $\partial A \bar{\partial} A$ Term

For the terms proportional to $\partial A \bar{\partial} Adz \wedge d\bar{z}$, since e^{-A} shows up both $\text{tr}(\mathbf{R}^2)$'s in $\text{tr}(\mathbf{R}^2) \wedge \text{tr}(\mathbf{R}^2)$ and one needs to integrate as in Eq. (2.12), the “factorization” do not occur as on the contrary to the case of $\partial\bar{\partial}A$ term. After some classification summarized in Appendix B, we obtain

$$\begin{aligned} & \int_{\text{Cone}_{\Sigma \rtimes I, n}} \mathcal{P}_{\text{anom}}^{7d, \text{double}}|_{\partial A \bar{\partial} A} \\ &= 32\pi \epsilon \int_{\Sigma \rtimes I} \left[2\mathbf{R}_N^2 \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) - 4\mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 - 4(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\hat{D}\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) \right. \\ & \quad \left. + 4(\hat{D}\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\hat{D}\mathbf{K} \cdot \bar{\mathbf{K}}) + 4(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{K}) + (\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge \text{tr}(\mathbf{r}^2) \right]. \end{aligned} \quad (3.9)$$

By summing the above two contributions, we obtain

$$32\pi \epsilon \int_{\Sigma \rtimes I} d \left[\mathbf{\Gamma}_N \wedge \mathbf{R}_N^2 + \frac{1}{2} \mathbf{\Gamma}_N \wedge \text{tr}(\mathbf{r}^2) + 2 \left[-\mathbf{R}_N + (\bar{\mathbf{K}} \cdot \mathbf{K}) \right] \wedge [(\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}}) + (\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K})] \right]. \quad (3.10)$$

Therefore, the holographic entanglement entropy formula for 7d double-trace gravitational

Chern-Simons term is obtained from the anomaly polynomial as

$$\begin{aligned}
S_{\text{ent,CS}}^{7d, \text{double}} &= 2^2(2\pi)(4) \int_{\Sigma} \left[\mathbf{\Gamma}_N \wedge \mathbf{R}_N^2 + \frac{1}{2} \mathbf{\Gamma}_N \wedge \text{tr}(\mathbf{r}^2) \right. \\
&\quad \left. + 2 \left[-\mathbf{R}_N + (\bar{\mathbf{K}} \cdot \mathbf{K}) \right] \wedge [(\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}}) + (\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K})] \right].
\end{aligned} \tag{3.11}$$

4 Direct Derivation from Chern-Simons Term

The purpose of this section is to reproduce (some of) the holographic entanglement entropy formulae obtained in §3 by directly applying the argument of [3] and [8] to the gravitational Chern-Simons terms, instead of the anomaly polynomials. Since the gravitational Chern-Simons terms depend on the connection one-form explicitly, we will first discuss the extension and modification of Dong’s formula in [8] to incorporate the Chern-Simons terms. As mentioned briefly in §§2.5, in the Chern-Simons-term-based derivation of the holographic entanglement entropy formulae, one needs to take into account a new type of order- ϵ contribution, which, as we will explain, never shows up in the anomaly-polynomial-based computation. To explain this type of terms through a concrete example, we will provide a brief review of the 3d case worked out by [42]. After this, we will present the computations for the 7d gravitational Chern-Simons terms and reproduce the result we obtained in §3. We will also explain why this new type of contribution never shows up nontrivially in the anomaly polynomials based computation in §3.

4.1 Holographic Entanglement Entropy for Chern-Simons Term

Let us first write down the generalization of Dong’s holographic entanglement entropy formula to a general purely gravitational Chern-Simons term and then give explanation of each term. For a general $(d+1)$ -dimensional purely gravitational Chern-Simons term $\mathbf{I}_{\text{CS}}[\mathbf{\Gamma}, \mathbf{R}]$ (d : an

positive even integer), the formal holographic entanglement entropy formula is given by¹⁵

$$\begin{aligned}
S_{\text{ent,CS}} &= S_{\text{ent}}^{\text{Wald}} + S_{\text{ent}}^{\text{Wald,extra}} + S_{\text{ent}}^{KK} + S_{\text{ent}}^{KK,\text{extra}} \quad \text{where} \\
S_{\text{ent}}^{\text{Wald}} &\equiv (2\pi) \int_{\Sigma} \left(\frac{\partial \mathbf{I}_{\text{CS}}}{\partial \mathbf{R}^z_z} \right) - (z \leftrightarrow \bar{z}), \\
S_{\text{ent}}^{\text{Wald,extra}} &\equiv (2\pi) \int_{\Sigma} \left(\frac{\partial \mathbf{I}_{\text{CS}}}{\partial \mathbf{\Gamma}^z_z} \right) \Big|_{K=\bar{K}=0, \text{integrate by part}} - (z \leftrightarrow \bar{z}), \\
S_{\text{ent}}^{KK} &\equiv +(8\pi) \int_{\Sigma} \mathbf{K}^i \left[\int_0^1 dt \, t \left(\frac{\partial^2 \mathbf{I}_{\text{CS}}}{\partial \mathbf{R}^i_z \partial \mathbf{R}^z_j} \right) \right]_{t \downarrow} \bar{\mathbf{K}}_j - (z \leftrightarrow \bar{z}) \\
&\quad - (4\pi) \int_{\Sigma} \mathbf{K}^i \left[\int_0^1 \frac{dt}{t} \left(\frac{\partial^2 \mathbf{I}_{\text{CS}}}{\partial \mathbf{R}^i_z \partial \mathbf{R}^j_{\bar{z}}} \right) \right]_{t \downarrow} \bar{\mathbf{K}}^j - (z \leftrightarrow \bar{z}), \\
S_{\text{ent}}^{KK,\text{extra}} &\equiv -(8\pi) \int_{\Sigma} \left[\int_0^1 dt \, t \left(\frac{\partial^2 \mathbf{I}_{\text{CS}}}{\partial \mathbf{\Gamma}^z_z \partial \mathbf{R}^z_j} \right) \right]_{t \downarrow} \bar{\mathbf{K}}_j - (z \leftrightarrow \bar{z}) \\
&\quad + (4\pi) \int_{\Sigma} \left[\int_0^1 \frac{dt}{t} \left(\frac{\partial^2 \mathbf{I}_{\text{CS}}}{\partial \mathbf{\Gamma}^z_z \partial \mathbf{R}^j_{\bar{z}}} \right) \right]_{t \downarrow} \bar{\mathbf{K}}^j - (z \leftrightarrow \bar{z}) \\
&\quad + (4\pi) \int_{\Sigma} \left[\int_0^1 \frac{dt}{t} \left(\frac{\partial^2 \mathbf{I}_{\text{CS}}}{\partial \mathbf{\Gamma}^z_z \partial \mathbf{\Gamma}^{\bar{z}}_{\bar{z}}} \right) \right]_{t \downarrow}. \tag{4.1}
\end{aligned}$$

In the above formula, essentially, the classification of the terms is done based on the origin of the singularities that show up in the Chern-Simons term on the regularized cone background. Comments and explanations of each term are in order:

- The terms $S_{\text{ent}}^{\text{Wald}}$ and S_{ent}^{KK} are those appearing in Dong's formula of [8]: $S_{\text{ent}}^{\text{Wald}}$ originates from the $\partial \bar{\partial} A$ terms in the curvature two-form \mathbf{R} , while S_{ent}^{KK} is from the $\partial A \bar{\partial} A$ terms whose ∂A and $\bar{\partial} A$ are both coming from \mathbf{R} . Therefore these terms only involve the derivatives of the Chern-Simons term with respect to \mathbf{R} .
- The term $S_{\text{ent}}^{KK,\text{extra}}$ originates in the $\partial A \bar{\partial} A$ terms where at least one of the ∂A and $\bar{\partial} A$ is from the connection one-form $\mathbf{\Gamma}$.
- In the evaluation of $S_{\text{ent}}^{\text{Wald}}$, we first take a derivative with respect to \mathbf{R} and then substitute ⁽⁰⁾ terms in Eqs. (A.4) and (A.9) with $A = 0$.
- In S_{ent}^{KK} and $S_{\text{ent}}^{KK,\text{extra}}$, we have introduced the integral with respect to a parameter t . The evaluation of these terms is carried out as follows. First, calculate the derivatives of the Chern-Simons terms, substitute all ⁽⁰⁾ terms in Eqs. (A.4) and (A.9) setting

¹⁵Here are two remarks on the derivatives with respect to \mathbf{R}^μ_ν and $\mathbf{\Gamma}^\mu_\nu$ in this formula. For the derivative $\partial/\partial \mathbf{R}^\mu_\nu$, all components of \mathbf{R}^μ_ν are treated as independent. For example, $\partial \mathbf{R}^z_i / \partial \mathbf{R}^i_z = 0$. This is purely conventional. We also note that the derivative $\partial/\partial \mathbf{\Gamma}^\mu_\nu$ acts like an exterior derivative operator (this is because $\mathbf{\Gamma}^\mu_\nu$ is a one-form). That is, for a wedge product of a p -form \mathbf{A} with a q -form \mathbf{B} , we have

$$\frac{\partial}{\partial \mathbf{\Gamma}^\mu_\nu} (\mathbf{A} \wedge \mathbf{B}) = \left(\frac{\partial \mathbf{A}}{\partial \mathbf{\Gamma}^\mu_\nu} \right) \wedge \mathbf{B} + (-1)^p \mathbf{A} \wedge \left(\frac{\partial \mathbf{B}}{\partial \mathbf{\Gamma}^\mu_\nu} \right).$$

$e^{-2A} = 1$. Then replace all \mathbf{K}_i and $\bar{\mathbf{K}}_j$ by $t^{1/2}\mathbf{K}_i$ and $t^{1/2}\bar{\mathbf{K}}_j$ respectively. Finally, do the integral with respect to t . This is a convenient way to capture the parameter ‘ q_α ’ of [8], which is introduced to count half the number of extrinsic curvatures in the integrands. The origin of this parameter is the integral (2.12) which needs to be carried out to compute the contributions from $\partial A \bar{\partial} A$ term.

- The term $S_{\text{ent}}^{\text{Wald, extra}}$ is the new type of contribution special to the Chern-Simons-term-based computation that we mentioned briefly in §§2.5. As we will soon explain in §§4.3, this type of terms do not generate any non-trivial extrinsic curvature dependent contribution. In addition, as explained in Eq. (2.14), either ∂A or $\bar{\partial} A$ needs to be supplied from $\mathbf{\Gamma}$ or \mathbf{R} . Moreover, since ∂A and $\bar{\partial} A$ are accompanied by the extrinsic curvatures in \mathbf{R} , this one derivative of A should originate in $\mathbf{\Gamma}$.

In the following, we provide a way to compute this type of terms through some simple examples. For 3d gravitational Chern-Simons term $\mathbf{I}_{\text{CS}}^{3d} = \text{tr}(\mathbf{\Gamma}\mathbf{R} - (1/3)\mathbf{\Gamma}^3) = \text{tr}(\mathbf{\Gamma}d\mathbf{\Gamma} + (2/3)\mathbf{\Gamma}^3)$, we first evaluate the derivative of the Chern-Simons term and pick up the terms which contain $d\mathbf{\Gamma}$:

$$(2\pi) \int_{\Sigma} \left(\frac{\partial \mathbf{I}_{\text{CS}}}{\partial \mathbf{\Gamma}_z^z} \right) \Big|_{K=\bar{K}=0} - (z \leftrightarrow \bar{z}) = (2\pi) \int_{\Sigma} (d\mathbf{\Gamma}_z^z) - (z \leftrightarrow \bar{z}). \quad (4.2)$$

The reason to keep the terms with $d\mathbf{\Gamma}$ is that one needs to integrate it by part as in Eq. (2.14). After this, we get rid of d of $\mathbf{\Gamma}_z^z$ (and its complex conjugate) and substitute $^{(0)}$ terms in Eqs. (A.9) and (A.4) with $A = 0$ and $\mathbf{K} = \bar{\mathbf{K}} = 0$. We then finally obtain

$$S_{\text{ent}}^{\text{Wald, extra}} = (2\pi) \int_{\Sigma} \left(\frac{\partial \mathbf{I}_{\text{CS}}}{\partial \mathbf{\Gamma}_z^z} \right) \Big|_{K=\bar{K}=0, \text{integration by part}} - (z \leftrightarrow \bar{z}) = (4\pi) \int_{\Sigma} \mathbf{\Gamma}_N. \quad (4.3)$$

This removal of d corresponds to the integration by part of ∂ or $\bar{\partial}$ explained in Eq. (2.14). More generally, the same procedure gives

$$\begin{aligned} \text{For } \mathcal{P}_{\text{anom}} &= \text{tr}(\mathbf{R}^{2k}), & S_{\text{ent}}^{\text{Wald, extra}} &= (4\pi) \int_{\Sigma} \mathbf{\Gamma}_N \wedge \mathbf{R}_N^{2k-2}, \\ \text{For } \mathcal{P}_{\text{anom}} &= \text{tr}(\mathbf{R}^{2k}) \wedge \text{tr}(\mathbf{R}^{2l}), \\ S_{\text{ent}}^{\text{Wald, extra}} &= (4\pi) \int_{\Sigma} \left(\mathbf{\Gamma}_N \wedge \mathbf{R}_N^{2k-2} \right) \wedge \text{tr}(\mathbf{R}^{2l}) \Big|_{0th, K=\bar{K}=0, A=0} \\ &\quad + (4\pi) \int_{\Sigma} \left(\mathbf{\Gamma}_N \wedge \mathbf{R}_N^{2l-2} \right) \wedge \text{tr}(\mathbf{R}^{2k}) \Big|_{0th, K=\bar{K}=0, A=0}. \end{aligned} \quad (4.4)$$

In the second line, $\text{tr}(\mathbf{R}^{2l}) \Big|_{0th, K=\bar{K}=0, A=0}$ means that $\text{tr}(\mathbf{R}^{2l})$ is evaluated by substituting $^{(0)}$ terms in Eqs. (A.9) with the extrinsic curvatures and A set to zero.

In the next subsections, in the course of the evaluation of 3d and 7d multi-trace gravitational Chern-Simons terms on the regularized cone geometry, we shall keep $\partial A, \bar{\partial} A$ and $\partial \bar{\partial} A$ all explicit such that the origin of each term in the formula will become clearer.

We note that, as we will explain in §5.1, generalization of Eq. (4.1) to a general mixed $U(1)$ -gravitational Chern-Simons term is straightforward since $U(1)$ field strength evaluated on the regularized cone does not generate extra singularities to be taken into account. That is, this formula (4.1) is still valid for the mixed Chern-Simons term.

4.2 3d Gravitational Chern-Simons Term

Let us first review the derivation of holographic entanglement entropy for the 3d gravitational Chern-Simons term which is worked out by [42]. From the expressions computed in Appendix A.5.7, naively, one might think that the nontrivial contribution at order ϵ comes only from $\mathbf{I}_{\text{CS}}^{3d}|_{\partial\bar{\partial}A}$, which after integration becomes

$$\int_{\text{Cone}_{\Sigma,n}} \mathbf{I}_{\text{CS}}^{3d}|_{\partial\bar{\partial}A} = \int_{\Sigma} 4\pi\epsilon \, \mathbf{\Gamma}_N. \quad (4.5)$$

However, as pointed out in [42] (see also [35]), a little more careful treatment is needed for $\mathbf{I}_{\text{CS}}^{3d}|_{\partial A, \bar{\partial}A}$. That is, the integration-by-part of the form (2.14) generates new type of contribution. Since we have already reviewed the rationale around Eq. (2.14), here we shall illustrate explicitly in this example what one gets from this term. In the current case, this term comes from

$$\int_{\text{Cone}_{\Sigma,n}} ({}^{(d)}\mathbf{\Gamma}_z^z) \wedge d\mathbf{\Gamma}_z^z = \int_{\text{Cone}_{\Sigma,n}} (-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge \mathbf{\Gamma}_z^z + \dots, \quad (4.6)$$

where we have applied Eq. (2.14). There is also a similar term coming from the integration by part of $({}^{(d)}\mathbf{\Gamma}_{\bar{z}}^{\bar{z}}) \wedge d\mathbf{\Gamma}_{\bar{z}}^{\bar{z}}$.

Altogether, after integration, what we get is

$$\int_{\text{Cone}_{\Sigma,n}} \mathbf{I}_{\text{CS}}^{3d}|_{\partial A, \bar{\partial}A} = \int_{\text{Cone}_{\Sigma,n}} (-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge \mathbf{\Gamma}_N \times 2 = \int_{\Sigma} 4\pi\epsilon \, \mathbf{\Gamma}_N. \quad (4.7)$$

Therefore, we finally obtain the holographic entanglement entropy formula for 3d Chern-Simons term as

$$S_{\text{ent,CS}}^{3d} = 8\pi \int_{\Sigma} \mathbf{\Gamma}_N. \quad (4.8)$$

This result by [42] is consistent with what we obtained by starting with the anomaly polynomial $\mathcal{P}_{\text{anom}}^{3d}$ in §3.

4.3 Remark on Integration-by-Part Terms

As we have seen in Eq. (4.6) for 3d gravitational Chern-Simons term, the integration-by-part under an appropriate regularization can give non-trivial contribution to the holographic entanglement entropy formula. Before discussing 7d case, here we explain some property of this “integration-by-part” type of terms.

The first remark is that this type of contribution does not depend on extrinsic curvatures. The reason is as follows: in the Chern-Simons term evaluated at the regularized cone geometry,

the extrinsic-curvature-dependent terms are always accompanied by the factor e^{-CA} with nonzero positive constant C . Then when we consider “integration-by-part” type of terms, the integration by part gives

$$\int dzd\bar{z}e^{-CA}(\partial A)d\Gamma(\dots) = - \int dzd\bar{z}e^{-CA}(\partial\bar{\partial}A - C\partial A\bar{\partial}A)\Gamma(\dots) + \dots = 0, \quad (4.9)$$

and similarly for $\bar{\partial}A$. Here we have used $\partial\bar{\partial}A = -\pi\epsilon\delta^2(z, \bar{z})$ and the integration of the form (2.12).

Another remark is that, when one evaluates the anomaly polynomial, we did not take into account this type of contribution. This can be justified as follows. When the anomaly polynomial is evaluated on the regularized cone background, since the terms proportional to $\partial A dz$ and $\bar{\partial} A d\bar{z}$ in \mathbf{R} are always accompanied by \mathbf{K}_i and/or $\bar{\mathbf{K}}_i$, these “integration-by-part” terms appearing in the anomaly polynomial are always proportional to e^{-CA} with nonzero C . Thus, from the above argument, this type of terms never contributes to holographic entanglement entropy formula so far as one start with the anomaly polynomials. On the other hand, when starting with the Chern-Simons terms, this type of term gives nontrivial contribution and thus we need to keep track of them carefully.

4.4 7d Double-Trace Gravitational Chern-Simons Term

Now we return to the direct derivation of holographic entanglement entropy formula from Chern-Simons terms. Here we consider the 7d double-trace type of gravitational Chern-Simons term $I_{\text{CS}}^{7d, \text{double}}$. We divide into the following three possible contribution and evaluate one by one :

$$\begin{aligned} \text{(Case 1)} \quad & I_{\text{CS}}^{3d}|_{\text{singular}} \wedge \mathcal{P}_{\text{anom}}^{3d}|_{0th}, \\ \text{(Case 2)} \quad & I_{\text{CS}}^{3d}|_{0th} \wedge \mathcal{P}_{\text{anom}}^{3d}|_{\text{singular}}, \\ \text{(Case 3)} \quad & I_{\text{CS}}^{3d}|_{\text{singular}} \wedge \mathcal{P}_{\text{anom}}^{3d}|_{\text{singular}}. \end{aligned}$$

Within each of Case 1 and Case 2, there are three possibilities depending on whether the singular term is proportional to ∂A (or $\bar{\partial}A$), $\partial\bar{\partial}A$, or $\partial A\bar{\partial}A$. Here the contribution proportional to ∂A (or $\bar{\partial}A$) means the one giving nontrivial contribution due to the integration-by-part (2.14). On the other hand, in Case 3, we only need to consider $\partial A dz$ and $\bar{\partial} A d\bar{z}$ terms in $I_{\text{CS}}^{3d}|_{\text{singular}}$ and $\text{tr}(\mathbf{R}^2)|_{\text{singular}}$, which after multiplication gives terms proportional to $\partial A\bar{\partial}A dz \wedge d\bar{z}$. We note that all the ingredients needed for this evaluation is summarized in Appendix A.

4.4.1 Case 1

We first consider Case 1. As mentioned above, Case 1 contains the following three possibilities:

(1) $\partial\bar{\partial}A$ Term

Since the this type of terms essentially “factorizes” into the computation of $\partial\bar{\partial}A$ term in 3d

gravitational Chern-Simons term and the non-singular term of the anomaly polynomial for 3d Chern-Simons term, we can easily obtain

$$\begin{aligned} & \int_{\text{Cone}_{\Sigma,n}} \mathbf{I}_{\text{CS}}^{3d}|_{\partial\bar{\partial}A} \wedge \mathcal{P}_{\text{anom}}^{3d}|_{0th} \\ &= \int_{\Sigma} 4\pi\epsilon \left[2\mathbf{R}_N^2 + \text{tr}(\mathbf{r}^2) - 8\mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) - 8(\hat{D}\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) + 8(\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{K}) \right] \wedge \Gamma_N. \end{aligned} \quad (4.10)$$

(2) $\partial A \bar{\partial} A$ Term

This type of contribution vanishes because $\mathbf{I}_{\text{CS}}^{3d}|_{\partial A \bar{\partial} A} = 0$:

$$\int_{\text{Cone}_{\Sigma,n}} \mathbf{I}_{\text{CS}}^{3d}|_{\partial A \bar{\partial} A} \wedge \mathcal{P}_{\text{anom}}^{3d}|_{0th} = 0. \quad (4.11)$$

(3) ∂A or $\bar{\partial} A$ Term

By integrating by part as in Eq. (2.14), we can see that this type of contribution gives nontrivial result:

$$\int_{\text{Cone}_{\Sigma,n}} \mathbf{I}_{\text{CS}}^{3d}|_{\partial A, \bar{\partial} A} \wedge \mathcal{P}_{\text{anom}}^{3d}|_{0th} = \int_{\Sigma} 4\pi\epsilon \left[2\Gamma_N \wedge \mathbf{R}_N^2 + \Gamma_N \wedge \text{tr}(\mathbf{r}^2) \right]. \quad (4.12)$$

4.4.2 Case 2

As in Case 1, we can evaluate the three possibilities in Case 2 one by one as follows :

(1) $\partial \bar{\partial} A$ Term

By using some results in Appendix A, we can straightforwardly obtain the following result:

$$\begin{aligned} & \int_{\text{Cone}_{\Sigma,n}} \mathbf{I}_{\text{CS}}^{3d}|_{0th} \wedge \mathcal{P}_{\text{anom}}^{3d}|_{\partial\bar{\partial}A} \\ &= \int_{\Sigma} (8\pi\epsilon) \left[2\Gamma_N \wedge \mathbf{R}_N^2 - 4\Gamma_N \wedge \mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) + \mathbf{R}_N \wedge \text{tr} \left(\gamma \mathbf{r} - \frac{1}{3} \gamma^3 \right) \right. \\ & \quad \left. - 4\mathbf{R}_N \wedge [(\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}}) + (\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K})] \right. \\ & \quad \left. - 2(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge \text{tr} \left(\gamma \mathbf{r} - \frac{1}{3} \gamma^3 \right) + 8(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge [(\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}}) + (\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K})] \right]. \end{aligned} \quad (4.13)$$

(2) $\partial A \bar{\partial} A$ Term

In the same way as $\partial \bar{\partial} A$ term above, by using the results in Appendix A, the direct evaluation gives

$$\begin{aligned} & \int_{\text{Cone}_{\Sigma,n}} \mathbf{I}_{\text{CS}}^{3d}|_{0th} \wedge \mathcal{P}_{\text{anom}}^{3d}|_{\partial A \bar{\partial} A} \\ &= \int_{\Sigma} 16\pi\epsilon \left[2\Gamma_N \wedge \mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) + (\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge \text{tr} \left(\gamma \mathbf{r} - \frac{1}{3} \gamma^3 \right) \right. \\ & \quad \left. - 2(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge [(\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}}) + (\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K})] \right]. \end{aligned} \quad (4.14)$$

(3) ∂A or $\bar{\partial} A$ Term

After integration by part as in Eq. (2.14), we can confirm that this type of term does not generate any nontrivial contribution :

$$\int_{\text{Cone}_{\Sigma,n}} \mathbf{I}_{\text{CS}}^{3d}|_{\text{0th}} \wedge \mathcal{P}_{\text{anom}}^{3d}|_{\partial A, \bar{\partial} A} = 0. \quad (4.15)$$

4.4.3 Case 3

By using some results in Appendix A, we can easily obtain

$$\begin{aligned} & \int_{\text{Cone}_{\Sigma,n}} \mathbf{I}_{\text{CS}}^{3d}|_{\partial A, \bar{\partial} A} \wedge \mathcal{P}_{\text{anom}}^{3d}|_{\partial A, \bar{\partial} A} \\ &= \int_{\Sigma} (-16\pi\epsilon) \left[\mathbf{R}_N \wedge [(\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) + (\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}})] - 2(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge [(\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) + (\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}})] \right]. \end{aligned} \quad (4.16)$$

4.4.4 Final Result

Summing results from all these cases, we obtain the total contribution as

$$\begin{aligned} & 32\pi\epsilon \int_{\Sigma} \left\{ \mathbf{\Gamma}_N \wedge \mathbf{R}_N^2 + \frac{1}{2} \mathbf{\Gamma}_N \wedge \text{tr}(\mathbf{r}^2) + 2[-\mathbf{R}_N + (\bar{\mathbf{K}} \cdot \mathbf{K})] \wedge [(\hat{D}\bar{\mathbf{K}} \cdot \mathbf{K}) + (\hat{D}\mathbf{K} \cdot \bar{\mathbf{K}})] \right. \\ & \quad \left. + d \left[\frac{1}{4} \mathbf{\Gamma}_N \wedge \text{tr} \left(\gamma \mathbf{r} - \frac{1}{3} \gamma^3 \right) + \frac{1}{2} \mathbf{\Gamma}_N \wedge [(\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}}) + (\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K})] \right] \right\}. \end{aligned}$$

Therefore, by neglecting the total derivative term (we assume that there is no homological obstruction), we finally obtain the holographic entanglement entropy formula for 7d double-trace gravitational Chern-Simons term as follows:

$$\begin{aligned} & S_{\text{ent,CS}}^{7d, \text{double}} \\ &= 32\pi \int_{\Sigma} \left\{ \mathbf{\Gamma}_N \wedge \mathbf{R}_N^2 + \frac{1}{2} \mathbf{\Gamma}_N \wedge \text{tr}(\mathbf{r}^2) + 2[-\mathbf{R}_N + (\bar{\mathbf{K}} \cdot \mathbf{K})] \wedge [(\hat{D}\bar{\mathbf{K}} \cdot \mathbf{K}) + (\hat{D}\mathbf{K} \cdot \bar{\mathbf{K}})] \right\}, \end{aligned} \quad (4.17)$$

which agrees with the result from the anomaly polynomial in §3.

A similar computation can be repeated for the single trace gravitational CS term in AdS₇

(i.e. Eq. (2.2)). The details are provided in Appendix D and the result is

$$\begin{aligned}
& \left[\int_{\text{Cone}_{\Sigma, n}} \mathbf{I}_{\text{CS}}^{7d \text{ single}} \right]_{\epsilon} \\
&= (16\pi\epsilon) \int_{\Sigma} \left[\mathbf{\Gamma}_N \wedge \mathbf{R}_N^2 + [-2\mathbf{R}_N + 3(\bar{\mathbf{K}} \cdot \mathbf{K})] \wedge [(\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}}) + (\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K})] \right. \\
&\quad + (\hat{D}\mathbf{K} \cdot \mathbf{r} \cdot \bar{\mathbf{K}}) - (\hat{D}\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{K}) \\
&\quad + d \left[\frac{1}{2} \mathbf{\Gamma}_N \wedge [(\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) + (\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}})] - \frac{3}{20} \mathbf{\Gamma}_N \wedge [(\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K}) + (\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}})] \right. \\
&\quad \left. \left. + \frac{1}{4} [(\hat{D}\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K}) - (\hat{D}\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}})] - \frac{3}{20} [(\bar{\mathbf{K}} \cdot \gamma^2 \cdot \mathbf{K}) + (\mathbf{K} \cdot \gamma^2 \cdot \bar{\mathbf{K}})] \right] \right]. \tag{4.18}
\end{aligned}$$

Neglecting the total derivative term (we assume that there is no homological obstruction), we finally obtain the holographic entanglement entropy formula for 7d single-trace gravitational Chern-Simons term as follows:

$$\begin{aligned}
S_{\text{ent,CS}}^{7d, \text{single}} &= (16\pi) \int_{\Sigma} \left[\mathbf{\Gamma}_N \wedge \mathbf{R}_N^2 + [-2\mathbf{R}_N + 3(\bar{\mathbf{K}} \cdot \mathbf{K})] \wedge [(\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}}) + (\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K})] \right. \\
&\quad \left. + (\hat{D}\mathbf{K} \cdot \mathbf{r} \cdot \bar{\mathbf{K}}) - (\hat{D}\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{K}) \right], \tag{4.19}
\end{aligned}$$

which agrees with the result from the anomaly polynomial in §3.

5 Consistency Checks and Applications

Now that we have obtained the holographic entanglement entropy formulae of purely gravitational Chern-Simons terms, we next generalize to some mixed Chern-Simons terms. After this, we will explain the consistency of our formulae with the black hole entropy formula for Chern-Simons terms. We will also comment on the Lorentz-frame dependence of Chern-Simons contribution to (holographic) entanglement entropy. At the end of this section, we will briefly comment on the application of our holographic entanglement entropy formula to 6d (2, 0) theories.

5.1 Generalization to Mixed Chern-Simons Term

Let us consider the cases with $U(1)$ -gravitational mixed Chern-Simons terms. In these cases, the corresponding anomaly polynomials contain the $U(1)$ field strength. Here we follow the notation in the Appendix of [8] and explain a straightforward generalization of holographic entanglement entropy formulae to the mixed Chern-Simons terms. The most general $U(1)$ gauge potential one-form $\mathbf{A}_{U(1)}$ on B_n (Here B_n is the bulk extension of n -replicated boundary

geometry M_n . We use $(\tilde{\rho}, \tilde{\tau}, y^i)$ for the coordinates of B_n can be written as

$$\begin{aligned} \mathbf{A}_{U(1)} = d\tilde{\rho} [a_\rho(y) + \mathcal{O}(\tilde{\rho}^2, \tilde{\rho}^n e^{\pm i n \tilde{\tau}})] + d\tilde{\tau} \tilde{\rho} [a_\tau(y) + \mathcal{O}(\tilde{\rho}^2, \tilde{\rho}^n e^{\pm i n \tilde{\tau}})] \\ + dy^i [a_i(y) + \mathcal{O}(\tilde{\rho}^2, \tilde{\rho}^n e^{\pm i n \tilde{\tau}})] . \end{aligned} \quad (5.1)$$

By introducing the new coordinate (ρ, τ, y^i) (which is a natural coordinate after the orbifolding by \mathbb{Z}_n) defined by $\tilde{\rho} = n\rho^{1/n}$ and $\tilde{\tau} = \tau/n$, we can rewrite this as

$$\mathbf{A}_{U(1)} = \rho^{-\epsilon} d\rho [a_\rho(y) + \mathcal{O}(z, \bar{z})] + \rho^{-\epsilon} \rho d\tau [a_\tau(y) + \mathcal{O}(z, \bar{z})] + dy^i [a_i(y) + \mathcal{O}(z, \bar{z})] . \quad (5.2)$$

Here $\epsilon = 1 - (1/n)$ and $(z, \bar{z}) = (\rho e^{i\tau}, \rho e^{-i\tau})$. Now we carry out the regularization of this cone geometry by introducing $A = -(\epsilon/2) \log(\rho^2 + a_{reg}^2)$ as

$$\mathbf{A}_{U(1)} = e^A d\rho [a_\rho(y) + \mathcal{O}(z, \bar{z})] + e^A \rho d\tau [a_\tau(y) + \mathcal{O}(z, \bar{z})] + dy^i [a_i(y) + \mathcal{O}(z, \bar{z})] . \quad (5.3)$$

From this gauge potential one-form on the regularized cone, we can compute the $U(1)$ field strength as

$$\begin{aligned} \mathbf{F}_{U(1)} &= d\mathbf{A}_{U(1)} \\ &= [\partial A dz \wedge d\bar{z} \times (\dots) + \bar{\partial} A dz \wedge d\bar{z} \times (\dots)] \\ &\quad + (\text{terms} \propto dz \text{ or } d\bar{z}) + \mathbf{F} + \mathcal{O}(z, \bar{z}) . \end{aligned} \quad (5.4)$$

Here the second line is the terms of order ϵ and the third line is of order one. In the third line we have also defined $\mathbf{F} = d(a_i(y) dy^i) = \partial_i a_j dy^i \wedge dy^j$. Combined with the observation in §2 (recall that only nontrivial contribution to the holographic entanglement entropy formulas comes from the terms proportional to $\partial \bar{\partial} A dz \wedge d\bar{z}$ or $\partial A \bar{\partial} A dz \wedge d\bar{z}$ in the anomaly polynomials), the order ϵ terms in $\mathbf{F}_{U(1)}$ do not generate any non-trivial contributions after the integration over (z, \bar{z}) , and only non-trivial contribution to the holographic entanglement entropy formula comes from the singular terms of \mathbf{R} -dependent part wedged with \mathbf{F} 's. In other words, whenever we start with the anomaly polynomials with \mathbf{F} , what we need to do is: (1) first replace $\mathbf{F}_{U(1)}$'s by \mathbf{F} , (2) for the gravitational part of the anomaly polynomial (i.e. $\mathbf{R}^\mu{}_\nu$ -dependent part), carry out the ϵ -expansion and extract the linear terms in ϵ -expansion. Therefore, once the purely gravitational cases are done, the mixed cases follow

almost automatically. For example, for a general positive integer k , we have

$$\text{AdS}_{2k+3} \text{ with } \mathcal{P}_{\text{anom}} = c \mathbf{F}^k \wedge \text{tr}(\mathbf{R}^2) \quad : \quad S_{\text{ent,CS}} = 2(2\pi)(2c) \int_{\Sigma} \mathbf{F}^k \wedge \Gamma_N, \quad (5.5)$$

$$\begin{aligned} & \text{AdS}_{2k+7} \text{ with } \mathcal{P}_{\text{anom}} = c \mathbf{F}^k \wedge \text{tr}(\mathbf{R}^4) \quad : \\ & S_{\text{ent,CS}} \\ & = 2(2\pi)(4c) \int_{\Sigma} \mathbf{F}^k \wedge \left[\Gamma_N \wedge \mathbf{R}_N^2 + [-2\mathbf{R}_N + 3(\bar{\mathbf{K}} \cdot \mathbf{K})] \wedge \left[(\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}}) + (\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) \right] \right. \\ & \quad \left. + (\hat{D}\mathbf{K} \cdot \mathbf{r} \cdot \bar{\mathbf{K}}) - (\hat{D}\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{K}) \right], \end{aligned} \quad (5.6)$$

$$\begin{aligned} & \text{AdS}_7 \text{ with } \mathcal{P}_{\text{anom}} = c \mathbf{F}^k \wedge \text{tr}(\mathbf{R}^2) \wedge \text{tr}(\mathbf{R}^2) \quad : \\ & S_{\text{ent,CS}} \\ & = 2^2(2\pi)(4c) \int_{\Sigma} \mathbf{F}^k \wedge \left[\Gamma_N \wedge \mathbf{R}_N^2 + \frac{1}{2} \Gamma_N \wedge \text{tr}(\mathbf{r}^2) \right. \\ & \quad \left. + 2[-\mathbf{R}_N + (\bar{\mathbf{K}} \cdot \mathbf{K})] \wedge \left[(\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}}) + (\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) \right] \right]. \end{aligned} \quad (5.7)$$

We note that the generalization to the mixed case with non-Abelian gauge field is straightforward: we can get the holographic entanglement entropy formula by multiplying the purely gravitational results with a wedge product of $\text{tr}(\mathbf{F}^k)$'s.

5.2 Consistency with Chern-Simons Contribution to Black Hole Entropy

For general stationary black holes, the extrinsic curvature vanishes at the bifurcation surface, $\mathbf{K}_i = 0$ and $\bar{\mathbf{K}}_i = 0$. We can then easily see that our holographic entanglement entropy for Chern-Simons terms evaluated there indeed reproduces the Tachikawa entropy formula

$$S_{\text{Tachikawa}} = \int_{\text{Bif}} \sum_{k=1}^{\infty} 8\pi k \Gamma_N (d\Gamma_N)^{2k-2} \frac{\partial \mathcal{P}_{\text{anom}}}{\partial \text{tr} \mathbf{R}^{2k}}. \quad (5.8)$$

We note that this formula was first proposed in [33] and then various covariance issues in its derivation were pointed out for 5d and higher dimensions in [34]. Recently, by using a manifestly covariant formulation of differential Noether charge, [38] gave a covariant derivation of this formula. In particular for the anomaly polynomials $\mathcal{P}_{\text{anom}}^{3d} = \text{tr}(\mathbf{R}^2)$, $\mathcal{P}_{\text{anom}}^{7d, \text{single}} = \text{tr}(\mathbf{R}^4)$ and $\mathcal{P}_{\text{anom}}^{7d, \text{double}} = \text{tr}(\mathbf{R}^2) \wedge \text{tr}(\mathbf{R}^2)$, the explicit forms of the entropy formulae are given

by¹⁶

$$S_{\text{Tachikawa}}^{3d} = 8\pi \int_{\text{Bif}} \mathbf{\Gamma}_N, \quad (5.9)$$

$$S_{\text{Tachikawa}}^{7d, \text{single}} = 2(8\pi) \int_{\text{Bif}} \mathbf{\Gamma}_N \wedge \mathbf{R}_N^2, \quad S_{\text{Tachikawa}}^{7d, \text{double}} = 2(8\pi) \int_{\text{Bif}} \mathbf{\Gamma}_N \wedge \text{tr}(\mathbf{R}^2). \quad (5.10)$$

We note that $\text{tr}(\mathbf{R}^2) = 2[\mathbf{R}_N^2 + (1/2)\text{tr}(\mathbf{r}^2)]$ which follows from Eq. (A.37) evaluated at the bifurcation surface. Therefore, the holographic entanglement entropy formulae with $\mathbf{K}_i = 0, \bar{\mathbf{K}}_i = 0$ at the bifurcation horizon indeed reproduce the Tachikawa entropy formula given as above at the bifurcation horizon. We note that the holographic entanglement entropy formulae for the mixed Chern-Simons terms derived in §§5.1 are also consistent with the Tachikawa entropy formula.

5.3 Lorentz Boost and Frame-Dependence

One of the interesting properties of the Chern-Simons contribution to entanglement entropy is that it depends on the choice of the Lorentz-frame. This reflects non-covariant transformation property of the Chern-Simons terms and quantum anomalies of Lorentz symmetry in the dual CFT side. A general local Lorentz boost can be decomposed into the ones tangent to the surface Σ and the ones normal to it. The holographic entanglement entropy formulae in Eq. (1.2)-(1.6) are obviously invariant under the Lorentz boosts tangent to Σ . We thus consider the normal one, written as a normal-bundle gauge transformation $\mathbf{\Gamma}_N \rightarrow \mathbf{\Gamma}_N + d\Lambda$. Under this transformation, the variation $\Delta S_{\text{ent,CS}}$ of the holographic entanglement entropy functionals is summarized as follows:

$$\text{AdS}_3 \text{ with } \mathcal{P}_{\text{anom}} = c \text{tr}(\mathbf{R}^2) \quad : \quad \Delta S_{\text{ent,CS}} = 2(2\pi)(2c) \int_{\partial\Sigma} \Lambda, \quad (5.11)$$

$$\text{AdS}_5 \text{ with } \mathcal{P}_{\text{anom}} = c \mathbf{F} \wedge \text{tr}(\mathbf{R}^2) \quad : \quad \Delta S_{\text{ent,CS}} = 2(2\pi)(2c) \int_{\partial\Sigma} \Lambda \mathbf{F}, \quad (5.12)$$

$$\text{AdS}_7 \text{ with } \mathcal{P}_{\text{anom}} = c \mathbf{F}^2 \wedge \text{tr}(\mathbf{R}^2) \quad : \quad \Delta S_{\text{ent,CS}} = 2(2\pi)(2c) \int_{\partial\Sigma} \Lambda \mathbf{F}^2, \quad (5.13)$$

$$\text{AdS}_7 \text{ with } \mathcal{P}_{\text{anom}} = c \text{tr}(\mathbf{R}^4) \quad : \quad \Delta S_{\text{ent,CS}} = 2(2\pi)(4c) \int_{\partial\Sigma} \Lambda \mathbf{R}_N^2, \quad (5.14)$$

$$\text{AdS}_7 \text{ with } \mathcal{P}_{\text{anom}} = c \text{tr}(\mathbf{R}^2) \wedge \text{tr}(\mathbf{R}^2) : \quad \Delta S_{\text{ent,CS}} = 2^2(2\pi)(4c) \int_{\partial\Sigma} \Lambda \left[\mathbf{R}_N^2 + \frac{1}{2} \text{tr}(\mathbf{r}^2) \right], \quad (5.15)$$

where $\partial\Sigma$ is the boundary of the bulk entangling surface Σ , which is identical to the entangling surface of the dual CFT. We note that we can trivially generalize this to more general mixed Chern-Simons terms discussed in §§5.1.

Here is a comment. For AdS_3 dual to CFT_2 , there is only one spatial direction normal to Σ . Therefore it is straightforward to generalize the above analysis to disconnected entangling

¹⁶In 3d, the black hole entropy formula below was in fact first derived in [35].

regions¹⁷ (i.e. multi-interval cases). For a m -interval vacuum entanglement entropy in CFT_2 , by doing the Lorentz boost with constant Λ , we obtain $\Delta S_{\text{ent,CS}} = (8\pi c)\Lambda(2m)$, where each interval contributes to a ‘2’ to the integral on the RHS of AdS_3 case above. This indeed reproduces the result obtained in [42, 49] for $m = 1$ as well as the Lorentz-boost-dependence in the multi-interval answer of [50] for CFTs in the large central charge limit.

As stated before, the main focus of the current paper is the derivation of holographic entanglement entropy formulae in the gravity side, but here we would like to comment on its relation to the dual CFT side. More generally, let us start with $(2k)$ -dimensional QFT (QFT_{2k} where k is a positive integer) with anomalies associated to some global continuous symmetries. The anomaly in this even-dimensional QFT can be understood systematically via the anomaly inflow mechanism [51] (see also [39] for finite temperature case). In the anomaly inflow mechanism, the anomalies observed in QFT_{2k} can be interpreted as follows: we regard that this even-dimensional QFT is living on the boundary of one-dimensional higher bulk (note that this bulk is irrelevant to gauge/gravity dualities) where an appropriate Chern-Simons term is turned on. This Chern-Simons term then generates a flow of current from the bulk to the boundary, explaining the breakdown of the conservation law due to the anomalies observed in this QFT_{2k} .

Once we have this description based on the anomaly-inflow mechanism, in order to compute anomalous contribution to entanglement entropy from QFT, what we need to do is to study the regularized cone geometry for this bulk Chern-Simons system of the anomaly-inflow. Since the Chern-Simons term in the anomaly inflow mechanism is exactly the same as the Chern-Simons term on the gravity side of gauge/gravity duality, the formal computation is exactly the same on both cases. Thus, in the same way as we have computed the holographic entanglement entropy formula from Chern-Simons term in §4, we can derive the anomalous contribution to entanglement entropy on QFT side. Then, by doing the $U(1)$ -gauge transformation along the normal bundle direction, we obtain the same result as we have obtained from the dual gravity side. Although we have used holography to obtain Eq. (5.11)-(5.15), the above argument suggests that the validity of these formulae for the frame-dependence part of entanglement entropy goes beyond CFTs with a gravity dual, and hence they are applicable for a generic QFT_{2k} with anomalies.

5.4 Application to Anomaly Polynomials of 6D (2, 0) theories

As an application, we derive the holographic entanglement entropy functional associated with anomaly polynomials of (2, 0) theories with gauge group G (see [52–54] for some detailed discussions):

$$\mathcal{P}_{\text{anom}} = -\frac{2\pi r_G}{48} \left[p_2(\mathbf{F}) - p_2(\mathbf{R}) + \frac{1}{4} (p_1(\mathbf{R}) - p_1(\mathbf{F}))^2 \right] - \frac{2\pi d_G h_G}{24} p_2(\mathbf{F}), \quad (5.16)$$

¹⁷See [49] for a nice geometric argument in the single interval case.

where r_G, d_G and h_G are the rank, the dimension and the Coxeter number of the gauge group G respectively while the Pontryagin classes are defined with the conventions:

$$p_1(\mathbf{F}) = \frac{1}{2} \left(\frac{i}{2\pi} \right)^2 \text{tr}(\mathbf{F}^2), \quad p_2(\mathbf{F}) = \frac{1}{8} \left(\frac{i}{2\pi} \right)^4 [(\text{tr} \mathbf{F}^2) \wedge (\text{tr} \mathbf{F}^2) - 2\text{tr}(\mathbf{F}^4)] , \quad (5.17)$$

and accordingly the same definitions for $p_1(\mathbf{R})$ and $p_2(\mathbf{R})$ upon sending $\mathbf{F} \rightarrow \mathbf{R}$. The field strength \mathbf{F} here is the $SO(5)_R$ field strength. Applying the results in Eqs. (5.5)-(5.7), we obtain

$$\begin{aligned} S_{\text{ent,CS}} &= \frac{r_G}{2 \times (48) \times (2\pi)^2} \int_{\Sigma} \left[-2\mathbf{\Gamma}_N \wedge \mathbf{R}_N^2 + \mathbf{\Gamma}_N \wedge [\text{tr}(\mathbf{r}^2) + \text{tr}(\mathbf{F}^2)] \right. \\ &\quad \left. + 4[\mathbf{R}_N - 2(\bar{\mathbf{K}} \cdot \mathbf{K})] \wedge [(\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) + (\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}})] \right. \\ &\quad \left. + 4[(\hat{D}\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{K}) - (\hat{D}\mathbf{K} \cdot \mathbf{r} \cdot \bar{\mathbf{K}})] \right] . \end{aligned} \quad (5.18)$$

6 Conclusion and Discussion

In this work, we have taken the first step towards understanding how extrinsic corrections to entanglement entropy can be reliably derived for gravitational theories with CS terms. While the Christoffel connection on the holographic replica background should be carefully dealt with, once this is done, one gets an answer that matches with indirect methods of computation using anomaly polynomial.

The simplest extension of our work would be to compute the extrinsic corrections for an arbitrary CS term and give a closed form answer. While this is in principle a straightforward exercise using the methods described in this paper, in practice, the computations become more and more tedious. Perhaps an intelligent reader can organise this computation into a nice closed form answer which uncovers the structure of these corrections.

Given the explicit form of corrections in the case of AdS_7 , there is a clear cut challenge to reproduce it from the field theory side by using CFT_6 with Lorentz anomalies. There is a good reason to assume that the corrections we have computed do not depend on the coupling (since they are tied to anomalies) and hence should be visible even in free theory computations. Both the free theory of chiral fermions and that of free chiral 2-form fields in 6d contribute to Lorentz anomalies and it is an interesting question as to how these extrinsic corrections show up in entanglement entropy computations of these theories. We hope that our work will motivate entanglement studies of these free theories which would shed light on the physics behind the Dong-type corrections.

Although we focused on the derivation of holographic entanglement entropy functional, in order to evaluate the Chern-Simons contribution to entanglement entropy for an entangling region for a given state, another step is involved—identification of the bulk entangling surface

in the dual gravitational background. For Einstein gravity, the location of the surface is identified in [3] by requiring that the singular terms in the bulk equation of motion vanish on the regularized cone background.¹⁸ This proves the prescription by Ryu and Takayanagi that the bulk entangling surface minimizes the holographic entanglement entropy functional.

For a general higher derivative theory, it is still unclear if the approach by [3], which becomes in general complicated, is equivalent to the minimization of the holographic entanglement functional (see [8, 15, 21, 22] for a related discussion in the context of Lovelock gravity). By using a relatively simple structure of the holographic entanglement entropy functional for Chern-Simons terms and bulk equation of motion, we hope that our analysis will lead to a deeper understanding of this second step. In [42], a geodesic equation for spinning particle was analyzed to identify the bulk entangling surface for 3d gravitational Chern-Simons term. It would be nice to generalize this approach to higher dimensions by considering spinning membranes.¹⁹

We note that the non-static situations might be necessary in order to get a nontrivial contribution to holographic entanglement entropy from the Chern-Simons terms. For the simplest case with the spherical entangling region for CFT vacuum state, the terms examined in this paper vanish. This follows from the fact that both the usual BH entropy part (given by Tachikawa formula) and the extrinsic curvature corrections vanish in this case.²⁰ It would be worthwhile to consider a gravitational background dual to an excited state or a more general entangling surface to obtain non-trivial contribution.

Another future direction is a generalization to time-dependent background to construct the CS analogue of HRT proposal [55] of Einstein gravity. Hydrodynamics for anomalous systems and its holographic description are of great interests both from theoretical and experimental point of view. As a result of recent developments, we now have a systematic understanding of the leading-order anomaly-induced transport both from quantum field theory side [39, 56] as well as dual gravity side [38, 40, 41]. It would be worthwhile to add entanglement entropy to the list of items to probe and characterize these anomalous systems.

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¹⁸See [21] for a more detailed analysis of the bulk replicated geometry.

¹⁹We thank A. Castro for interesting discussions on this issue.

²⁰We would like to thank Mukund Rangamani for emphasizing this point.

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A Some Useful Quantities

This Appendix summarizes the connection one-form $\mathbf{\Gamma}$, curvature two-form \mathbf{R} and their wedge products evaluated on the regularized cone background (2.7). We will use them in the evaluation of anomaly polynomials as well as gravitational Chern-Simons terms. It is noted that, as we mentioned in §2, we dropped many terms that are irrelevant to the evaluation of the holographic entanglement entropy formulas for Chern-Simons terms in this paper.

Here is one remark on the notation. In this Appendix, after evaluating \mathbf{R} etc. on the regularized cone geometry, we will decompose them into pieces and use the following notation and terminology: By superscript (0) (for example, $(0)\mathbf{R}$), we label the contributions of order- ϵ^0 and at the same time zeroth order in (z, \bar{z}) -expansion, while the superscript (d) labels the order- ϵ term of the form $\partial A dz$ or $\bar{\partial} A d\bar{z}$ multiplied by zeroth order terms in (z, \bar{z}) -expansion. By superscript (d^2) , we denote the order- ϵ term of the form $\partial \bar{\partial} A dz \wedge d\bar{z}$ multiplied by zeroth order terms in (z, \bar{z}) -expansion. Finally the superscript (dd) is used for the order- ϵ^2 term of the form $\partial A \bar{\partial} A dz \wedge d\bar{z}$ multiplied by zeroth order terms in (z, \bar{z}) -expansion.

A.1 Christoffel Symbol

On the regularized cone geometry (2.7), the Christoffel symbol up to order ϵ^1 and up to first order in (z, \bar{z}) -expansion is computed as

$$\begin{aligned}
\Gamma^z_{zz} &= 2\partial A - 4e^{2A}T\bar{z}, & \Gamma^z_{z\bar{z}} &= 2e^{2A}Tz & \Gamma^z_{\bar{z}\bar{z}} &= 0, \\
\Gamma^z_{zj} &= +2i\epsilon U_j - i(zV_{zj} + \bar{z}V_{\bar{z}j}) - 2i(U_j + V_{aj}x^a) + 2iU_m g^{ml}K_{zlj}z, \\
\Gamma^z_{\bar{z}j} &= 2iU_m g^{ml}K_{\bar{z}lj}z, \\
\Gamma^z_{jk} &= -iz(\partial_j U_k + \partial_k U_j) - 2e^{-2A}(K_{\bar{z}jk} + Q_{\bar{z}ajk}x^a) + 2iU_m \gamma^m_{jk}z, \\
\Gamma^{\bar{z}}_{\bar{z}\bar{z}} &= 2\bar{\partial}A - 4e^{2A}Tz, & \Gamma^{\bar{z}}_{z\bar{z}} &= 2e^{2A}T\bar{z}, & \Gamma^{\bar{z}}_{zz} &= 0, \\
\Gamma^{\bar{z}}_{\bar{z}j} &= -2i\epsilon U_j + i(zV_{zj} + \bar{z}V_{\bar{z}j}) + 2i(U_j + V_{aj}x^a) - 2iU_m g^{ml}K_{\bar{z}lj}\bar{z}, \\
\Gamma^{\bar{z}}_{zj} &= -2iU_m g^{ml}K_{zlj}\bar{z}, \\
\Gamma^{\bar{z}}_{jk} &= i\bar{z}(\partial_j U_k + \partial_k U_j) - 2e^{-2A}(K_{zjk} + Q_{zajk}x^a) - 2iU_m \gamma^m_{jk}\bar{z}, \\
\Gamma^i_{zz} &= ie^{2A}g^{il}V_{zl}\bar{z}, & \Gamma^i_{\bar{z}\bar{z}} &= -ie^{2A}g^{il}V_{\bar{z}l}z, \\
\Gamma^i_{z\bar{z}} &= -\frac{i}{2}e^{2A}g^{il}(V_{zl}z - V_{\bar{z}l}\bar{z}), \\
\Gamma^i_{jz} &= -2e^{2A}\bar{z}g^{il}U_l U_j + g^{il}K_{zjl} - 2g^{im}g^{ln}K_{amn}x^a K_{zjl} \\
&\quad + \frac{i}{2}e^{2A}\bar{z}g^{il}(\partial_j U_l - \partial_l U_j) + g^{il}Q_{zajl}x^a, \\
\Gamma^i_{j\bar{z}} &= -2e^{2A}zg^{il}U_l U_j + g^{il}K_{\bar{z}jl} - 2g^{im}g^{ln}K_{amn}x^a K_{\bar{z}jl} \\
&\quad - \frac{i}{2}e^{2A}zg^{il}(\partial_j U_l - \partial_l U_j) + g^{il}Q_{\bar{z}ajl}x^a, \\
\Gamma^i_{jk} &= -2g^{il}U_l(K_{zjk}z - K_{\bar{z}jk}\bar{z}) + \gamma^i_{jk} - 2g^{in}K_{anm}x^a \gamma^m_{jk} \\
&\quad + g^{il}(\partial_j K_{alk} + \partial_k K_{ajl} - \partial_l K_{ajk})x^a.
\end{aligned} \tag{A.1}$$

Here we have defined the Christoffel symbol for g_{ij} as $\gamma^i_{jk} \equiv 1/2g^{il}(\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk})$.

It is also convenient for our purpose to define the normal-bundle connection $(\Gamma_N)_i$ and its field strength as

$$(\Gamma_N)_i \equiv (-2iU_i), \quad (R_N)_{ij} \equiv 2\partial_{[i}(\Gamma_N)_{j]}, \tag{A.2}$$

with their differential forms being $\mathbf{\Gamma}_N \equiv (\Gamma_N)_i dx^i$ and $\mathbf{R}_N \equiv d\mathbf{\Gamma}_N$.

A.2 Connection One-Form

The connection one-form is defined by $\mathbf{\Gamma}^\mu_\nu \equiv \Gamma^\mu_{\nu\rho} dx^\rho$. From the expression of the Christoffel symbol in Eq. (A.1), it is obvious that, in the connection one-form, ∂A and $\bar{\partial}A$ are always accompanied by dz and $d\bar{z}$, respectively. Up to zeroth order in (z, \bar{z}) -expansion and up to order ϵ^1 , the connection one-form is computed as

$$\begin{aligned}
\mathbf{\Gamma}^z_z &= 2\partial A dz + \mathbf{\Gamma}_N + \dots, & \mathbf{\Gamma}^{\bar{z}}_{\bar{z}} &= 2\bar{\partial}A d\bar{z} - \mathbf{\Gamma}_N + \dots, & \mathbf{\Gamma}^z_{\bar{z}} &= \mathbf{\Gamma}^{\bar{z}}_z = 0 + \dots, \\
\mathbf{\Gamma}^z_j &= -2e^{-2A}\bar{\mathbf{K}}_j + \dots, & \mathbf{\Gamma}^{\bar{z}}_j &= -2e^{-2A}\mathbf{K}_j + \dots, \\
\mathbf{\Gamma}^i_z &= \mathbf{K}^i + \dots, & \mathbf{\Gamma}^i_{\bar{z}} &= \bar{\mathbf{K}}^i + \dots, & \mathbf{\Gamma}^i_j &= \gamma^i_j + \dots,
\end{aligned} \tag{A.3}$$

where we have defined extrinsic curvature one-forms $\mathbf{K}_i \equiv K_{zij} dy^j$, $\bar{\mathbf{K}}_i \equiv \bar{K}_{\bar{z}ij} dy^j$, $\mathbf{K}^i \equiv g^{il} K_{zlj} dy^j$, $\bar{\mathbf{K}}^i \equiv g^{il} K_{\bar{z}lj} dy^j$ and the connection one-form $\gamma^i_j \equiv \gamma^i_{jk} dy^k$ for γ^i_{jk} . For later convenience (in the direct evaluation from Chern-Simons terms in §4), here we decompose the above expression of the connection one form as follows:

$$\begin{aligned}
(d)\Gamma^z_z &= 2\partial A dz, & (d)\Gamma^{\bar{z}}_{\bar{z}} &= 2\bar{\partial} A d\bar{z}, & (d)\Gamma^z_{\bar{z}} &= 0, & (d)\Gamma^{\bar{z}}_z &= 0, \\
(d)\Gamma^z_j &= 0, & (d)\Gamma^{\bar{z}}_j &= 0, & (d)\Gamma^i_z &= 0, & (d)\Gamma^i_{\bar{z}} &= 0, & (d)\Gamma^i_j &= 0, \\
(0)\Gamma^z_z &= \Gamma_N, & (0)\Gamma^{\bar{z}}_{\bar{z}} &= -\Gamma_N, & (0)\Gamma^z_{\bar{z}} &= \Gamma^{\bar{z}}_z = 0, \\
(0)\Gamma^z_j &= (-2e^{-2A})\bar{\mathbf{K}}_j, & (0)\Gamma^{\bar{z}}_j &= (-2e^{-2A})\mathbf{K}_j, & (0)\Gamma^i_z &= \mathbf{K}^i, & (0)\Gamma^i_{\bar{z}} &= \bar{\mathbf{K}}^i, \\
(0)\Gamma^i_j &= \gamma^i_j.
\end{aligned} \tag{A.4}$$

We note that we have dropped the terms proportional to dz or $d\bar{z}$ without ∂A or $\bar{\partial} A$ accompanied, since they are irrelevant for the purpose of this paper as mentioned in §2. In the rest of this Appendix, we will also drop these terms.

A.3 Derivative of Connection One-Form

We next evaluate the derivative of the connection one-form, $d\Gamma^\mu_\nu$ up to order ϵ^1 and up to zeroth order in (z, \bar{z}) -expansion:

$$\begin{aligned}
d\Gamma^z_z &= -2\partial\bar{\partial} A dz \wedge d\bar{z} + \mathbf{R}_N + \dots, & d\Gamma^{\bar{z}}_{\bar{z}} &= 2\partial\bar{\partial} A dz \wedge d\bar{z} - \mathbf{R}_N + \dots, \\
d\Gamma^z_{\bar{z}} &= 0, & d\Gamma^{\bar{z}}_z &= 0 + \dots, & d\Gamma^i_z &= d\mathbf{K}^i + \dots, & d\Gamma^i_{\bar{z}} &= d\bar{\mathbf{K}}^i + \dots, \\
d\Gamma^z_j &= (-2e^{-2A})(-2dA) \wedge \bar{\mathbf{K}}_j + (-2e^{-2A})d\bar{\mathbf{K}}_j + \dots, \\
d\Gamma^{\bar{z}}_j &= (-2e^{-2A})(-2dA) \wedge \mathbf{K}_j + (-2e^{-2A})d\mathbf{K}_j + \dots, \\
d\Gamma^i_j &= d\gamma^i_j + \dots.
\end{aligned} \tag{A.5}$$

A.4 Curvature Two-Form

By using the above results, the curvature two-form $\mathbf{R}^\mu_\nu \equiv d\Gamma^\mu_\nu + \Gamma^\mu_\rho \wedge \Gamma^\rho_\nu$ is calculated as

$$\begin{aligned}
\mathbf{R}^z_z &= -2\partial\bar{\partial} A dz \wedge d\bar{z} + \mathbf{R}_N + (-2e^{-2A})(\bar{\mathbf{K}} \cdot \mathbf{K}) + \dots, \\
\mathbf{R}^{\bar{z}}_{\bar{z}} &= 2\partial\bar{\partial} A dz \wedge d\bar{z} - \mathbf{R}_N - (-2e^{-2A})(\bar{\mathbf{K}} \cdot \mathbf{K}) + \dots, \\
\mathbf{R}^z_{\bar{z}} &= \mathbf{R}^{\bar{z}}_z = 0, \\
\mathbf{R}^i_z &= -2\partial A dz \wedge \mathbf{K}^i + \hat{D}\mathbf{K}^i + \dots, \\
\mathbf{R}^i_{\bar{z}} &= -2\bar{\partial} A d\bar{z} \wedge \bar{\mathbf{K}}^i + \hat{D}\bar{\mathbf{K}}^i + \dots, \\
\mathbf{R}^z_j &= (-2e^{-2A})(-2\bar{\partial} A d\bar{z}) \wedge \bar{\mathbf{K}}_j + (-2e^{-2A})\hat{D}\bar{\mathbf{K}}_j + \dots, \\
\mathbf{R}^{\bar{z}}_j &= (-2e^{-2A})(-2\partial A dz) \wedge \mathbf{K}_j + (-2e^{-2A})\hat{D}\mathbf{K}_j + \dots, \\
\mathbf{R}^i_j &= \mathbf{r}^i_j - 2e^{-2A}(\mathbf{K}^i \wedge \bar{\mathbf{K}}_j + \bar{\mathbf{K}}^i \wedge \mathbf{K}_j) + \dots,
\end{aligned} \tag{A.6}$$

where $\bar{\mathbf{K}} \cdot \mathbf{K} \equiv \bar{\mathbf{K}}^i \wedge \mathbf{K}_i$. Here the covariant derivative \hat{D} compatible with the normal bundle gauge field Γ_N is defined such that

$$\begin{aligned}
\hat{D}\mathbf{K}_i &\equiv D\mathbf{K}_i - \Gamma_N \wedge \mathbf{K}_i, & \hat{D}\bar{\mathbf{K}}_i &\equiv D\bar{\mathbf{K}}_i + \Gamma_N \wedge \bar{\mathbf{K}}_i, \\
\hat{D}\mathbf{K}^i &\equiv D\mathbf{K}^i - \Gamma_N \wedge \mathbf{K}^i, & \hat{D}\bar{\mathbf{K}}^i &\equiv D\bar{\mathbf{K}}^i + \Gamma_N \wedge \bar{\mathbf{K}}^i,
\end{aligned} \tag{A.7}$$

where the covariant derivative D with respect to g_{ij} is defined as

$$\begin{aligned} DK_i &\equiv dK_i - \gamma^j_i \wedge K_j, & D\bar{K}_i &\equiv d\bar{K}_i - \gamma^j_i \wedge \bar{K}_j, \\ DK^i &\equiv dK^i + \gamma^i_j \wedge K^j, & D\bar{K}^i &\equiv d\bar{K}^i + \gamma^i_j \wedge \bar{K}^j. \end{aligned} \quad (\text{A.8})$$

We also note that $\mathbf{r}^i_j \equiv (1/2)r^i_{jkl}dy^k \wedge dy^l \equiv d\gamma^i_j + \gamma^i_k \wedge \gamma^k_j$ is the curvature two-form for g_{ij} .

In Eq. (A.6), we have written down the explicit form of the terms in the curvature two-form that are needed for our purpose. By using the notation introduced at the beginning of this Appendix, it is convenient to decompose the above expression in the following way:

$$\begin{aligned} (d^2)\mathbf{R}^z_z &= -2\partial\bar{\partial}Adz \wedge d\bar{z}, & (d^2)\mathbf{R}^{\bar{z}}_{\bar{z}} &= 2\partial\bar{\partial}Adz \wedge d\bar{z}, \\ (d^2)\mathbf{R}^z_{\bar{z}} &= 0, & (d^2)\mathbf{R}^{\bar{z}}_z &= 0, \\ (d^2)\mathbf{R}^i_z &= 0, & (d^2)\mathbf{R}^i_{\bar{z}} &= 0, & (d^2)\mathbf{R}^z_j &= 0, & (d^2)\mathbf{R}^{\bar{z}}_j &= 0, & (d^2)\mathbf{R}^i_j &= 0, \\ (dd)\mathbf{R}^\mu_\nu &= 0, \\ (d)\mathbf{R}^z_z &= 0, & (d)\mathbf{R}^{\bar{z}}_{\bar{z}} &= 0, & (d)\mathbf{R}^z_{\bar{z}} &= 0, & (d)\mathbf{R}^{\bar{z}}_z &= 0, \\ (d)\mathbf{R}^i_z &= -2\partial Adz \wedge \mathbf{K}^i, & (d)\mathbf{R}^i_{\bar{z}} &= -2\bar{\partial} Ad\bar{z} \wedge \bar{\mathbf{K}}^i, \\ (d)\mathbf{R}^z_j &= (-2e^{-2A})(-2\bar{\partial} Ad\bar{z}) \wedge \bar{\mathbf{K}}_j, & (d)\mathbf{R}^{\bar{z}}_j &= (-2e^{-2A})(-2\partial Adz) \wedge \mathbf{K}_j, \\ (d)\mathbf{R}^i_j &= 0, \\ (0)\mathbf{R}^z_z &= \mathbf{R}_N + (-2e^{-2A})(\bar{\mathbf{K}} \cdot \mathbf{K}), & (0)\mathbf{R}^{\bar{z}}_{\bar{z}} &= -\mathbf{R}_N - (-2e^{-2A})(\bar{\mathbf{K}} \cdot \mathbf{K}), \\ (0)\mathbf{R}^z_{\bar{z}} &= 0, & (0)\mathbf{R}^{\bar{z}}_z &= 0, \\ (0)\mathbf{R}^i_z &= \hat{D}\mathbf{K}^i, & (0)\mathbf{R}^i_{\bar{z}} &= \hat{D}\bar{\mathbf{K}}^i, \\ (0)\mathbf{R}^z_j &= (-2e^{-2A})\hat{D}\bar{\mathbf{K}}_j, & (0)\mathbf{R}^{\bar{z}}_j &= (-2e^{-2A})\hat{D}\mathbf{K}_j, \\ (0)\mathbf{R}^i_j &= \mathbf{r}^i_j + (-2e^{-2A})(\mathbf{K}^i \wedge \bar{\mathbf{K}}_j + \bar{\mathbf{K}}^i \wedge \mathbf{K}_j). \end{aligned} \quad (\text{A.9})$$

A.5 Structures of Products of Connections and Curvatures

A.5.1 Γ^2

For the wedge product $(\Gamma^2)^\mu_\nu = \Gamma^\mu_\rho \wedge \Gamma^\rho_\nu$, after some computation, we can obtain the following result:

$$(d^2)(\Gamma^2)^\mu_\nu = 0, \quad (dd)(\Gamma^2)^\mu_\nu = 0, \quad (\text{A.10})$$

$$\begin{aligned} (d)(\Gamma^2)^z_z &= 0, & (d)(\Gamma^2)^{\bar{z}}_{\bar{z}} &= 0, & (d)(\Gamma^2)^z_{\bar{z}} &= 0, & (d)(\Gamma^2)^{\bar{z}}_z &= 0, \\ (d)(\Gamma^2)^z_j &= (-2e^{-2A})(2\partial Adz) \wedge \bar{\mathbf{K}}_j, & (d)(\Gamma^2)^{\bar{z}}_j &= (-2e^{-2A})(2\bar{\partial} Ad\bar{z}) \wedge \mathbf{K}_j, \\ (d)(\Gamma^2)^i_z &= \mathbf{K}^i \wedge (2\partial Adz), & (d)(\Gamma^2)^i_{\bar{z}} &= \bar{\mathbf{K}}^i \wedge (2\bar{\partial} Ad\bar{z}), \\ (d)(\Gamma^2)^i_j &= 0, \end{aligned} \quad (\text{A.11})$$

and

$$\begin{aligned}
{}^{(0)}(\Gamma^2)^z{}_z &= (-2e^{-2A})(\bar{\mathbf{K}} \cdot \mathbf{K}), & {}^{(0)}(\Gamma^2)^{\bar{z}}{}_{\bar{z}} &= -(-2e^{-2A})(\bar{\mathbf{K}} \cdot \mathbf{K}), \\
{}^{(0)}(\Gamma^2)^z{}_{\bar{z}} &= 0, & {}^{(0)}(\Gamma^2)^{\bar{z}}{}_z &= 0, \\
{}^{(0)}(\Gamma^2)^z{}_j &= (-2e^{-2A}) [\Gamma_N \wedge \bar{\mathbf{K}}_j + (\bar{\mathbf{K}} \cdot \gamma)_j], \\
{}^{(0)}(\Gamma^2)^{\bar{z}}{}_j &= (-2e^{-2A}) [-\Gamma_N \wedge \mathbf{K}_j + (\mathbf{K} \cdot \gamma)_j], \\
{}^{(0)}(\Gamma^2)^i{}_z &= \mathbf{K}^i \wedge \Gamma_N + (\gamma \cdot \mathbf{K})^i, & {}^{(0)}(\Gamma^2)^i{}_{\bar{z}} &= -\bar{\mathbf{K}}^i \wedge \Gamma_N + (\gamma \cdot \bar{\mathbf{K}})^i, \\
{}^{(0)}(\Gamma^2)^i{}_j &= (-2e^{-2A})(\mathbf{K}^i \wedge \bar{\mathbf{K}}_j + \bar{\mathbf{K}}^i \wedge \mathbf{K}_j) + (\gamma^2)^i{}_j.
\end{aligned} \tag{A.12}$$

Here $(\bar{\mathbf{K}} \cdot \gamma)_j = \bar{\mathbf{K}}_i \wedge \gamma^i{}_j$, $(\gamma^2)^i{}_j = \gamma^i{}_k \wedge \gamma^k{}_j$ etc.

A.5.2 Γ^3

The wedge product Γ^3 is decomposed as follows:

$${}^{(d^2)}(\Gamma^3)^\mu{}_\nu = 0, \quad {}^{(dd)}(\Gamma^3)^\mu{}_\nu = 0, \tag{A.13}$$

$$\begin{aligned}
{}^{(d)}(\Gamma^3)^z{}_z &= 2(2\partial Adz)(-2e^{-2A})(\bar{\mathbf{K}} \cdot \mathbf{K}), & {}^{(d)}(\Gamma^3)^{\bar{z}}{}_{\bar{z}} &= (-2)(2\bar{\partial} Ad\bar{z})(-2e^{-2A})(\bar{\mathbf{K}} \cdot \mathbf{K}), \\
{}^{(d)}(\Gamma^3)^z{}_{\bar{z}} &= 0, & {}^{(d)}(\Gamma^3)^{\bar{z}}{}_z &= 0, \\
{}^{(d)}(\Gamma^3)^z{}_j &= (-2e^{-2A})(2\partial Adz) \wedge (\bar{\mathbf{K}} \cdot \gamma)_j, & {}^{(d)}(\Gamma^3)^{\bar{z}}{}_j &= (-2e^{-2A})(2\bar{\partial} Ad\bar{z}) \wedge (\mathbf{K} \cdot \gamma)_j, \\
{}^{(d)}(\Gamma^3)^i{}_z &= (2\partial Adz) \wedge (\gamma \cdot \mathbf{K})^i, \\
{}^{(d)}(\Gamma^3)^i{}_{\bar{z}} &= (2\bar{\partial} Ad\bar{z}) \wedge (\gamma \cdot \bar{\mathbf{K}})^i, \\
{}^{(d)}(\Gamma^3)^i{}_j &= (2\partial Adz)(-2e^{-2A})(-1)(\mathbf{K}^i \wedge \bar{\mathbf{K}}_j) + (2\bar{\partial} Ad\bar{z})(-2e^{-2A})(-1)(\bar{\mathbf{K}}^i \wedge \mathbf{K}_j),
\end{aligned} \tag{A.14}$$

and

$$\begin{aligned}
{}^{(0)}(\Gamma^3)^z{}_z &= (-2e^{-2A}) [2\Gamma_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) + (\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K})], \\
{}^{(0)}(\Gamma^3)^{\bar{z}}{}_{\bar{z}} &= (-2e^{-2A}) [2\Gamma_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) + (\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}})], \\
{}^{(0)}(\Gamma^3)^z{}_{\bar{z}} &= (-2e^{-2A})(\bar{\mathbf{K}} \cdot \gamma \cdot \bar{\mathbf{K}}), & {}^{(0)}(\Gamma^3)^{\bar{z}}{}_z &= (-2e^{-2A})(\mathbf{K} \cdot \gamma \cdot \mathbf{K}), \\
{}^{(0)}(\Gamma^3)^z{}_j &= (-2e^{-2A})^2(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge \bar{\mathbf{K}}_j + (-2e^{-2A}) [\Gamma_N \wedge (\bar{\mathbf{K}} \cdot \gamma)_j + (\bar{\mathbf{K}} \cdot \gamma \cdot \gamma)_j], \\
{}^{(0)}(\Gamma^3)^{\bar{z}}{}_j &= -(-2e^{-2A})^2(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge \mathbf{K}_j + (-2e^{-2A}) [-\Gamma_N \wedge (\mathbf{K} \cdot \gamma)_j + (\mathbf{K} \cdot \gamma \cdot \gamma)_j], \\
{}^{(0)}(\Gamma^3)^i{}_z &= [(\gamma \cdot \mathbf{K})^i \wedge \Gamma_N + (\gamma \cdot \gamma \cdot \mathbf{K})^i] + (-2e^{-2A})\mathbf{K}^i \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}), \\
{}^{(0)}(\Gamma^3)^i{}_{\bar{z}} &= [-(\gamma \cdot \bar{\mathbf{K}})^i \wedge \Gamma_N + (\gamma \cdot \gamma \cdot \bar{\mathbf{K}})^i] - (-2e^{-2A})\bar{\mathbf{K}}^i \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}), \\
{}^{(0)}(\Gamma^3)^i{}_j &= (\gamma^3)^i{}_j \\
&\quad + (-2e^{-2A}) [-\Gamma_N \wedge (\mathbf{K}^i \wedge \bar{\mathbf{K}}_j - \bar{\mathbf{K}}^i \wedge \mathbf{K}_j) \\
&\quad + (\gamma \cdot \mathbf{K})^i \wedge \bar{\mathbf{K}}_j + (\gamma \cdot \bar{\mathbf{K}})^i \wedge \mathbf{K}_j + \mathbf{K}^i \wedge (\bar{\mathbf{K}} \cdot \gamma)_j + \bar{\mathbf{K}}^i \wedge (\mathbf{K} \cdot \gamma)_j].
\end{aligned} \tag{A.15}$$

A.5.3 ΓR

The wedge product ΓR can be computed as follows:

$$\begin{aligned}
(d^2)(\Gamma R)^z_z &= \Gamma_N \wedge (-2\partial\bar{\partial}Adz \wedge d\bar{z}), & (d^2)(\Gamma R)^{\bar{z}}_{\bar{z}} &= \Gamma_N \wedge (-2\partial\bar{\partial}Adz \wedge d\bar{z}), \\
(d^2)(\Gamma R)^z_{\bar{z}} &= 0, & (d^2)(\Gamma R)^{\bar{z}}_z &= 0, & (d^2)(\Gamma R)^z_j &= 0, & (d^2)(\Gamma R)^{\bar{z}}_j &= 0, \\
(d^2)(\Gamma R)^i_z &= \mathbf{K}^i \wedge (-2\partial\bar{\partial}Adz \wedge d\bar{z}), & (d^2)(\Gamma R)^i_{\bar{z}} &= (-1)\bar{\mathbf{K}}^i \wedge (-2\partial\bar{\partial}Adz \wedge d\bar{z}), \\
(d^2)(\Gamma R)^i_j &= 0, \\
(dd)(\Gamma R)^z_z &= 0, & (dd)(\Gamma R)^{\bar{z}}_{\bar{z}} &= 0, & (dd)(\Gamma R)^z_{\bar{z}} &= 0, & (dd)(\Gamma R)^{\bar{z}}_z &= 0, \\
(dd)(\Gamma R)^z_j &= (-2e^{-2A})(-4\partial A\bar{\partial}Adz \wedge d\bar{z}) \wedge \bar{\mathbf{K}}_j, & (dd)(\Gamma R)^{\bar{z}}_j &= (-2e^{-2A})(4\partial A\bar{\partial}Adz \wedge d\bar{z}) \wedge \mathbf{K}_j, \\
(dd)(\Gamma R)^i_z &= 0, & (dd)(\Gamma R)^i_{\bar{z}} &= 0, \\
(dd)(\Gamma R)^i_j &= 0,
\end{aligned} \tag{A.16}$$

and

$$\begin{aligned}
(d)(\Gamma R)^z_z &= (2\partial Adz) \wedge [\mathbf{R}_N + 2(-2e^{-2A})(\bar{\mathbf{K}} \cdot \mathbf{K})], \\
(d)(\Gamma R)^{\bar{z}}_{\bar{z}} &= -(2\bar{\partial} Ad\bar{z}) \wedge [\mathbf{R}_N + 2(-2e^{-2A})(\bar{\mathbf{K}} \cdot \mathbf{K})], \\
(d)(\Gamma R)^z_{\bar{z}} &= 0, & (d)(\Gamma R)^{\bar{z}}_z &= 0, \\
(d)(\Gamma R)^z_j &= (2\partial Adz) \wedge (-2e^{-2A})\hat{D}\bar{\mathbf{K}}_j + (-2e^{-2A})(2\bar{\partial} Ad\bar{z}) \wedge \Gamma_N \wedge \bar{\mathbf{K}}_j, \\
(d)(\Gamma R)^{\bar{z}}_j &= (2\bar{\partial} Ad\bar{z}) \wedge (-2e^{-2A})\hat{D}\mathbf{K}_j - (-2e^{-2A})(2\partial Adz) \wedge \Gamma_N \wedge \mathbf{K}_j, \\
(d)(\Gamma R)^i_z &= (2\partial Adz) \wedge (\gamma \cdot \mathbf{K})^i, & (d)(\Gamma R)^i_{\bar{z}} &= (2\bar{\partial} Ad\bar{z}) \wedge (\gamma \cdot \bar{\mathbf{K}})^i, \\
(d)(\Gamma R)^i_j &= (2\partial Adz) \wedge (-2e^{-2A})(\bar{\mathbf{K}}^i \wedge \mathbf{K}_j) + (2\bar{\partial} Ad\bar{z}) \wedge (-2e^{-2A})(\mathbf{K}^i \wedge \bar{\mathbf{K}}_j), \tag{A.17}
\end{aligned}$$

$$\begin{aligned}
(0)(\Gamma R)^z_z &= \Gamma_N \wedge \mathbf{R}_N + (-2e^{-2A}) \left[\Gamma_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) + \bar{\mathbf{K}} \cdot \hat{D}\mathbf{K} \right], \\
(0)(\Gamma R)^{\bar{z}}_{\bar{z}} &= \Gamma_N \wedge \mathbf{R}_N + (-2e^{-2A}) \left[\Gamma_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) + \mathbf{K} \cdot \hat{D}\bar{\mathbf{K}} \right], \\
(0)(\Gamma R)^z_{\bar{z}} &= (-2e^{-2A})(\bar{\mathbf{K}} \cdot \hat{D}\bar{\mathbf{K}}), \\
(0)(\Gamma R)^{\bar{z}}_z &= (-2e^{-2A})(\mathbf{K} \cdot \hat{D}\mathbf{K}), \\
(0)(\Gamma R)^z_j &= (-2e^{-2A}) \left[\Gamma_N \wedge \hat{D}\bar{\mathbf{K}}_j + (\bar{\mathbf{K}} \cdot \mathbf{r})_j \right] + (-2e^{-2A})^2(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge \bar{\mathbf{K}}_j, \\
(0)(\Gamma R)^{\bar{z}}_j &= (-2e^{-2A}) \left[-\Gamma_N \wedge \hat{D}\mathbf{K}_j + (\mathbf{K} \cdot \mathbf{r})_j \right] + (-1)(-2e^{-2A})^2(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge \mathbf{K}_j, \\
(0)(\Gamma R)^i_z &= \left[\mathbf{R}_N \wedge \mathbf{K}^i + (\gamma \cdot \hat{D}\mathbf{K})^i \right] + (-2e^{-2A})\mathbf{K}^i \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}), \\
(0)(\Gamma R)^i_{\bar{z}} &= \left[-\mathbf{R}_N \wedge \bar{\mathbf{K}}^i + (\gamma \cdot \hat{D}\bar{\mathbf{K}})^i \right] + (-1)(-2e^{-2A})\bar{\mathbf{K}}^i \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}), \\
(0)(\Gamma R)^i_j &= (\gamma \mathbf{r})^i_j \\
&\quad + (-2e^{-2A}) \left[\mathbf{K}^i \wedge \hat{D}\bar{\mathbf{K}}_j + \bar{\mathbf{K}}^i \wedge \hat{D}\mathbf{K}_j + (\gamma \cdot \mathbf{K})^i \wedge \bar{\mathbf{K}}_j + (\gamma \cdot \bar{\mathbf{K}})^i \wedge \mathbf{K}_j \right].
\end{aligned} \tag{A.18}$$

A.5.4 $R\Gamma$

After some straightforward computation, the wedge product $R\Gamma$ is obtained as

$$\begin{aligned}
(d^2)(R\Gamma)^z_z &= (-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge \Gamma_N, & (d^2)(R\Gamma)^{\bar{z}}_{\bar{z}} &= (-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge \Gamma_N, \\
(d^2)(R\Gamma)^z_{\bar{z}} &= 0, & (d^2)(R\Gamma)^{\bar{z}}_z &= 0, \\
(d^2)(R\Gamma)^z_j &= (-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge (-2e^{-2A})\bar{\mathbf{K}}_j, & (d^2)(R\Gamma)^{\bar{z}}_j &= (-1)(-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge (-2e^{-2A})\mathbf{K}_j, \\
(d^2)(R\Gamma)^i_z &= 0, & (d^2)(R\Gamma)^i_{\bar{z}} &= 0, & (d^2)(R\Gamma)^i_j &= 0, \\
(d^d)(R\Gamma)^\mu_\nu &= 0,
\end{aligned} \tag{A.19}$$

and

$$\begin{aligned}
(d)(R\Gamma)^z_z &= (2\partial Adz) \wedge [\mathbf{R}_N + (-2e^{-2A})(\bar{\mathbf{K}} \cdot \mathbf{K})] + (-2\bar{\partial} Ad\bar{z}) \wedge (-2e^{-2A})(\bar{\mathbf{K}} \cdot \mathbf{K}), \\
(d)(R\Gamma)^{\bar{z}}_{\bar{z}} &= (-2\bar{\partial} Ad\bar{z}) \wedge [\mathbf{R}_N + (-2e^{-2A})(\bar{\mathbf{K}} \cdot \mathbf{K})] + (2\partial Adz) \wedge (-2e^{-2A})(\bar{\mathbf{K}} \cdot \mathbf{K}), \\
(d)(R\Gamma)^z_{\bar{z}} &= 0, & (d)(R\Gamma)^{\bar{z}}_z &= 0, \\
(d)(R\Gamma)^z_j &= (-2\bar{\partial} Ad\bar{z}) \wedge (-2e^{-2A})(\bar{\mathbf{K}} \cdot \gamma)_j, & (d)(R\Gamma)^{\bar{z}}_j &= (-2\partial Adz) \wedge (-2e^{-2A})(\mathbf{K} \cdot \gamma)_j, \\
(d)(R\Gamma)^i_z &= (2\partial Adz) \wedge [\hat{D}\mathbf{K}^i - \mathbf{K}^i \wedge \Gamma_N], & (d)(R\Gamma)^i_{\bar{z}} &= (2\bar{\partial} Ad\bar{z}) \wedge [\hat{D}\bar{\mathbf{K}}^i + \bar{\mathbf{K}}^i \wedge \Gamma_N], \\
(d)(R\Gamma)^i_j &= (-2\partial Adz) \wedge (-2e^{-2A})(\mathbf{K}^i \wedge \bar{\mathbf{K}}_j) + (-2\bar{\partial} Ad\bar{z}) \wedge (-2e^{-2A})(\bar{\mathbf{K}}^i \wedge \mathbf{K}_j),
\end{aligned} \tag{A.20}$$

$$\begin{aligned}
(0)(R\Gamma)^z_z &= \Gamma_N \wedge \mathbf{R}_N + (-2e^{-2A})[\Gamma_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) + (\hat{D}\bar{\mathbf{K}} \cdot \mathbf{K})], \\
(0)(R\Gamma)^{\bar{z}}_{\bar{z}} &= \Gamma_N \wedge \mathbf{R}_N + (-2e^{-2A})[\Gamma_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) + (\hat{D}\mathbf{K} \cdot \bar{\mathbf{K}})], \\
(0)(R\Gamma)^z_{\bar{z}} &= (-2e^{-2A})(\hat{D}\bar{\mathbf{K}} \cdot \bar{\mathbf{K}}), & (0)(R\Gamma)^{\bar{z}}_z &= (-2e^{-2A})(\hat{D}\mathbf{K} \cdot \mathbf{K}), \\
(0)(R\Gamma)^z_j &= (-2e^{-2A}) \left[\mathbf{R}_N \wedge \bar{\mathbf{K}}_j + (\hat{D}\bar{\mathbf{K}} \cdot \gamma)_j \right] + (-2e^{-2A})^2(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge \bar{\mathbf{K}}_j, \\
(0)(R\Gamma)^{\bar{z}}_j &= (-2e^{-2A}) \left[-\mathbf{R}_N \wedge \mathbf{K}_j + (\hat{D}\mathbf{K} \cdot \gamma)_j \right] - (-2e^{-2A})^2(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge \mathbf{K}_j, \\
(0)(R\Gamma)^i_z &= [\hat{D}\mathbf{K}^i \wedge \Gamma_N + (\mathbf{r} \cdot \mathbf{K})^i] + (-2e^{-2A})\mathbf{K}^i \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}), \\
(0)(R\Gamma)^i_{\bar{z}} &= [-\hat{D}\bar{\mathbf{K}}^i \wedge \Gamma_N + (\mathbf{r} \cdot \bar{\mathbf{K}})^i] - (-2e^{-2A})\bar{\mathbf{K}}^i \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}), \\
(0)(R\Gamma)^i_j &= (\mathbf{r} \cdot \gamma)^i_j + (-2e^{-2A})[\hat{D}\mathbf{K}^i \wedge \bar{\mathbf{K}}_j + \hat{D}\bar{\mathbf{K}}^i \wedge \mathbf{K}_j + \mathbf{K}^i \wedge (\bar{\mathbf{K}} \cdot \gamma)_j + \bar{\mathbf{K}}^i \wedge (\mathbf{K} \cdot \gamma)_j].
\end{aligned} \tag{A.21}$$

A.5.5 $\Gamma^2 \mathbf{R}$

We can also compute the wedge product $\Gamma^2 \mathbf{R}$ as

$$\begin{aligned}
(d^2)(\Gamma^2 \mathbf{R})^z_z &= (-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge (-2e^{-2A})(\bar{\mathbf{K}} \cdot \mathbf{K}), \\
(d^2)(\Gamma^2 \mathbf{R})^{\bar{z}}_{\bar{z}} &= (-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge (-2e^{-2A})(\bar{\mathbf{K}} \cdot \mathbf{K}), \\
(d^2)(\Gamma^2 \mathbf{R})^z_{\bar{z}} &= 0, \quad (d^2)(\Gamma^2 \mathbf{R})^{\bar{z}}_z = 0, \quad (d^2)(\Gamma^2 \mathbf{R})^z_j = 0, \quad (d^2)(\Gamma^2 \mathbf{R})^{\bar{z}}_j = 0, \\
(d^2)(\Gamma^2 \mathbf{R})^i_z &= (-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge [\mathbf{K}^i \wedge \Gamma_N + (\gamma \cdot \mathbf{K})^i], \\
(d^2)(\Gamma^2 \mathbf{R})^i_{\bar{z}} &= (-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge [\bar{\mathbf{K}}^i \wedge \Gamma_N - (\gamma \cdot \bar{\mathbf{K}})^i], \\
(d^2)(\Gamma^2 \mathbf{R})^i_j &= 0, \\
(dd)(\Gamma^2 \mathbf{R})^z_z &= 0, \quad (dd)(\Gamma^2 \mathbf{R})^{\bar{z}}_{\bar{z}} = 0, \quad (dd)(\Gamma^2 \mathbf{R})^z_{\bar{z}} = 0, \quad (dd)(\Gamma^2 \mathbf{R})^{\bar{z}}_z = 0, \\
(dd)(\Gamma^2 \mathbf{R})^z_j &= 0, \quad (dd)(\Gamma^2 \mathbf{R})^{\bar{z}}_j = 0, \quad (dd)(\Gamma^2 \mathbf{R})^i_z = 0, \quad (dd)(\Gamma^2 \mathbf{R})^i_{\bar{z}} = 0, \\
(dd)(\Gamma^2 \mathbf{R})^i_j &= (-4\partial A \bar{\partial} Adz \wedge d\bar{z}) \wedge (-2e^{-2A})(\mathbf{K}^i \wedge \bar{\mathbf{K}}_j - \bar{\mathbf{K}}^i \wedge \mathbf{K}_j), \tag{A.22}
\end{aligned}$$

and

$$\begin{aligned}
(d)(\Gamma^2 \mathbf{R})^z_z &= (2\partial Adz) \wedge (-2e^{-2A}) \left[(\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) - \Gamma_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) - (\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K}) \right], \\
(d)(\Gamma^2 \mathbf{R})^{\bar{z}}_{\bar{z}} &= (2\bar{\partial} Ad\bar{z}) \wedge (-2e^{-2A}) \left[(\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}}) - \Gamma_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) - (\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}}) \right], \\
(d)(\Gamma^2 \mathbf{R})^z_{\bar{z}} &= (2\partial Adz) \wedge (-2e^{-2A})(\bar{\mathbf{K}} \cdot \hat{D}\bar{\mathbf{K}}) + (-2\bar{\partial} Ad\bar{z}) \wedge (-2e^{-2A})(\bar{\mathbf{K}} \cdot \gamma \cdot \bar{\mathbf{K}}), \\
(d)(\Gamma^2 \mathbf{R})^{\bar{z}}_z &= (2\bar{\partial} Ad\bar{z}) \wedge (-2e^{-2A})(\mathbf{K} \cdot \hat{D}\mathbf{K}) + (-2\partial Adz) \wedge (-2e^{-2A})(\mathbf{K} \cdot \gamma \cdot \mathbf{K}), \\
(d)(\Gamma^2 \mathbf{R})^z_j &= (2\partial Adz) \wedge [(-2e^{-2A})(\bar{\mathbf{K}} \cdot \mathbf{r})_j + (-2e^{-2A})^2(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge \bar{\mathbf{K}}_j] \\
&\quad + (-2\bar{\partial} Ad\bar{z}) \wedge (-2e^{-2A})^2(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge \bar{\mathbf{K}}_j, \\
(d)(\Gamma^2 \mathbf{R})^{\bar{z}}_j &= (2\bar{\partial} Ad\bar{z}) \wedge [(-2e^{-2A})(\mathbf{K} \cdot \mathbf{r})_j - (-2e^{-2A})^2(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge \mathbf{K}_j] \\
&\quad + (2\partial Adz) \wedge (-2e^{-2A})^2(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge \mathbf{K}_j, \\
(d)(\Gamma^2 \mathbf{R})^i_z &= (-2\partial Adz) \wedge [\mathbf{R}_N \wedge \mathbf{K}^i + (\gamma \cdot \gamma \cdot \mathbf{K})^i + 2(-2e^{-2A})(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge \mathbf{K}^i], \\
(d)(\Gamma^2 \mathbf{R})^i_{\bar{z}} &= (-2\bar{\partial} Ad\bar{z}) \wedge [-\mathbf{R}_N \wedge \bar{\mathbf{K}}^i + (\gamma \cdot \gamma \cdot \bar{\mathbf{K}})^i - 2(-2e^{-2A})(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge \bar{\mathbf{K}}^i], \\
(d)(\Gamma^2 \mathbf{R})^i_j &= (-2\partial Adz) \wedge (-2e^{-2A}) \left[(\mathbf{K}^i \wedge \hat{D}\bar{\mathbf{K}}_j) + [-\bar{\mathbf{K}}^i \wedge \Gamma_N + (\gamma \cdot \bar{\mathbf{K}})^i] \wedge \mathbf{K}_j \right] \\
&\quad + (-2\bar{\partial} Ad\bar{z}) \wedge (-2e^{-2A}) \left[(\bar{\mathbf{K}}^i \wedge \hat{D}\mathbf{K}_j) + [\mathbf{K}^i \wedge \Gamma_N + (\gamma \cdot \mathbf{K})^i] \wedge \bar{\mathbf{K}}_j \right], \tag{A.23}
\end{aligned}$$

$$\begin{aligned}
^{(0)}(\Gamma^2 \mathbf{R})^z_z &= (-2e^{-2A})[\mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) + \Gamma_N \wedge (\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) + (\bar{\mathbf{K}} \cdot \gamma \cdot \hat{D}\mathbf{K})] + (-2e^{-2A})^2(\bar{\mathbf{K}} \cdot \mathbf{K})^2, \\
^{(0)}(\Gamma^2 \mathbf{R})^{\bar{z}}_{\bar{z}} &= (-2e^{-2A})[\mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) - \Gamma_N \wedge (\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}}) + (\mathbf{K} \cdot \gamma \cdot \hat{D}\bar{\mathbf{K}})] + (-2e^{-2A})^2(\bar{\mathbf{K}} \cdot \mathbf{K})^2, \\
^{(0)}(\Gamma^2 \mathbf{R})^z_{\bar{z}} &= (-2e^{-2A}) \left[\Gamma_N \wedge (\bar{\mathbf{K}} \cdot \hat{D}\bar{\mathbf{K}}) + (\bar{\mathbf{K}} \cdot \gamma \cdot \hat{D}\bar{\mathbf{K}}) \right], \\
^{(0)}(\Gamma^2 \mathbf{R})^{\bar{z}}_z &= (-2e^{-2A}) \left[-\Gamma_N \wedge (\mathbf{K} \cdot \hat{D}\mathbf{K}) + (\mathbf{K} \cdot \gamma \cdot \hat{D}\mathbf{K}) \right], \\
^{(0)}(\Gamma^2 \mathbf{R})^z_j &= (-2e^{-2A})[\Gamma_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{r})_j + (\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{r})_j] \\
&\quad + (-2e^{-2A})^2 \left[(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge [\hat{D}\bar{\mathbf{K}}_j + \Gamma_N \wedge \bar{\mathbf{K}}_j] + (\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K}) \wedge \bar{\mathbf{K}}_j + (\bar{\mathbf{K}} \cdot \gamma \cdot \bar{\mathbf{K}}) \wedge \mathbf{K}_j \right], \\
^{(0)}(\Gamma^2 \mathbf{R})^{\bar{z}}_j &= (-2e^{-2A})[-\Gamma_N \wedge (\mathbf{K} \cdot \mathbf{r})_j + (\mathbf{K} \cdot \gamma \cdot \mathbf{r})_j] \\
&\quad + (-2e^{-2A})^2 \left[-(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge [\hat{D}\mathbf{K}_j - \Gamma_N \wedge \mathbf{K}_j] + (\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}}) \wedge \mathbf{K}_j + (\mathbf{K} \cdot \gamma \cdot \mathbf{K}) \wedge \bar{\mathbf{K}}_j \right], \\
^{(0)}(\Gamma^2 \mathbf{R})^i_z &= [\mathbf{K}^i \wedge \Gamma_N + (\gamma \cdot \mathbf{K})^i] \wedge \mathbf{R}_N + (\gamma \cdot \gamma \cdot \hat{D}\mathbf{K})^i \\
&\quad + (-2e^{-2A}) \left[[\mathbf{K}^i \wedge \Gamma_N + (\gamma \cdot \mathbf{K})^i] \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) + \mathbf{K}^i \wedge (\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) + \bar{\mathbf{K}}^i \wedge (\mathbf{K} \cdot \hat{D}\mathbf{K}) \right], \\
^{(0)}(\Gamma^2 \mathbf{R})^i_{\bar{z}} &= [\bar{\mathbf{K}}^i \wedge \Gamma_N - (\gamma \cdot \bar{\mathbf{K}})^i] \wedge \mathbf{R}_N + (\gamma \cdot \gamma \cdot \hat{D}\bar{\mathbf{K}})^i \\
&\quad + (-2e^{-2A}) \left[[\bar{\mathbf{K}}^i \wedge \Gamma_N - (\gamma \cdot \bar{\mathbf{K}})^i] \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) + \bar{\mathbf{K}}^i \wedge (\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}}) + \mathbf{K}^i \wedge (\bar{\mathbf{K}} \cdot \hat{D}\bar{\mathbf{K}}) \right], \\
^{(0)}(\Gamma^2 \mathbf{R})^i_j &= (\gamma \cdot \gamma \cdot \mathbf{r})^i_j \\
&\quad + (-2e^{-2A}) \left[[\mathbf{K}^i \wedge \Gamma_N + (\gamma \cdot \mathbf{K})^i] \wedge \hat{D}\bar{\mathbf{K}}_j + [-\bar{\mathbf{K}}^i \wedge \Gamma_N + (\gamma \cdot \bar{\mathbf{K}})^i] \wedge \hat{D}\mathbf{K}_j \right. \\
&\quad \left. + \mathbf{K}^i \wedge (\bar{\mathbf{K}} \cdot \mathbf{r})_j + \bar{\mathbf{K}}^i \wedge (\mathbf{K} \cdot \mathbf{r})_j + (\gamma \cdot \gamma \cdot \mathbf{K})^i \wedge \bar{\mathbf{K}}_j + (\gamma \cdot \gamma \cdot \bar{\mathbf{K}})^i \wedge \mathbf{K}_j \right] \\
&\quad + (-2e^{-2A})^2(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\mathbf{K}^i \wedge \bar{\mathbf{K}}_j - \bar{\mathbf{K}}^i \wedge \mathbf{K}_j).
\end{aligned} \tag{A.24}$$

A.5.6 \mathbf{R}^2

For \mathbf{R}^2 , after some computation, we obtain the following result:

$$\begin{aligned}
^{(d^2)}(\mathbf{R}^2)^z_z &= 2(-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge [\mathbf{R}_N + (-2e^{-2A})(\bar{\mathbf{K}} \cdot \mathbf{K})], \\
^{(d^2)}(\mathbf{R}^2)^{\bar{z}}_{\bar{z}} &= 2(-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge [\mathbf{R}_N + (-2e^{-2A})(\bar{\mathbf{K}} \cdot \mathbf{K})], \\
^{(d^2)}(\mathbf{R}^2)^z_{\bar{z}} &= 0, \quad ^{(d^2)}(\mathbf{R}^2)^{\bar{z}}_z = 0, \\
^{(d^2)}(\mathbf{R}^2)^z_j &= (-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge (-2e^{-2A})\hat{D}\bar{\mathbf{K}}_j \\
^{(d^2)}(\mathbf{R}^2)^{\bar{z}}_j &= (-1)(-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge (-2e^{-2A})\hat{D}\mathbf{K}_j, \\
^{(d^2)}(\mathbf{R}^2)^i_z &= (-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge \hat{D}\mathbf{K}^i, \\
^{(d^2)}(\mathbf{R}^2)^i_{\bar{z}} &= (-1)(-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge \hat{D}\bar{\mathbf{K}}^i, \\
^{(d^2)}(\mathbf{R}^2)^i_j &= 0,
\end{aligned} \tag{A.25}$$

and

$$\begin{aligned}
^{(dd)}(\mathbf{R}^2)^z_z &= (4\partial A\bar{\partial}Adz \wedge d\bar{z}) \wedge (-2e^{-2A})(\bar{\mathbf{K}} \cdot \mathbf{K}), \\
^{(dd)}(\mathbf{R}^2)^{\bar{z}}_{\bar{z}} &= (4\partial A\bar{\partial}Adz \wedge d\bar{z}) \wedge (-2e^{-2A})(\bar{\mathbf{K}} \cdot \mathbf{K}), \\
^{(dd)}(\mathbf{R}^2)^z_{\bar{z}} &= 0, \quad ^{(dd)}(\mathbf{R}^2)^{\bar{z}}_z = 0, \quad ^{(dd)}(\mathbf{R}^2)^z_j = 0, \quad ^{(dd)}(\mathbf{R}^2)^{\bar{z}}_j = 0, \\
^{(dd)}(\mathbf{R}^2)^i_z &= 0, \quad ^{(dd)}(\mathbf{R}^2)^i_{\bar{z}} = 0, \\
^{(dd)}(\mathbf{R}^2)^i_j &= (-1)(-2e^{-2A})(4\partial A\bar{\partial}Adz \wedge d\bar{z}) \wedge (\mathbf{K}^i \wedge \bar{\mathbf{K}}_j - \bar{\mathbf{K}}^i \wedge \mathbf{K}_j), \tag{A.26}
\end{aligned}$$

as well as

$$\begin{aligned}
^{(d)}(\mathbf{R}^2)^z_z &= (-2e^{-2A})(-2\partial Adz) \wedge (\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}}) \\
&\quad + (-2e^{-2A})(-2\bar{\partial}Ad\bar{z}) \wedge (\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}), \\
^{(d)}(\mathbf{R}^2)^{\bar{z}}_{\bar{z}} &= (-2e^{-2A})(-2\partial Adz) \wedge (\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}}) \\
&\quad + (-2e^{-2A})(-2\bar{\partial}Ad\bar{z}) \wedge (\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}), \\
^{(d)}(\mathbf{R}^2)^z_{\bar{z}} &= 2(-2e^{-2A})(-2\bar{\partial}Ad\bar{z})(\bar{\mathbf{K}} \cdot \hat{D}\bar{\mathbf{K}}), \\
^{(d)}(\mathbf{R}^2)^{\bar{z}}_z &= 2(-2e^{-2A})(-2\partial Adz)(\mathbf{K} \cdot \hat{D}\mathbf{K}), \\
^{(d)}(\mathbf{R}^2)^z_j &= (-2\bar{\partial}Ad\bar{z}) \wedge [(-2e^{-2A})(\bar{\mathbf{K}} \cdot \mathbf{r})_j + \mathbf{R}_N \wedge \bar{\mathbf{K}}_j] + 2(-2e^{-2A})^2(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge \bar{\mathbf{K}}_j, \\
^{(d)}(\mathbf{R}^2)^{\bar{z}}_j &= (-2\partial Adz) \wedge [(-2e^{-2A})(\mathbf{K} \cdot \mathbf{r})_j - \mathbf{R}_N \wedge \mathbf{K}_j] - 2(-2e^{-2A})^2(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge \mathbf{K}_j, \\
^{(d)}(\mathbf{R}^2)^i_z &= (-2\partial Adz) \wedge [\mathbf{K}^i \wedge \mathbf{R}_N + (\mathbf{r} \cdot \mathbf{K})^i + 2(-2e^{-2A})\mathbf{K}^i \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})], \\
^{(d)}(\mathbf{R}^2)^i_{\bar{z}} &= (-2\bar{\partial}Ad\bar{z}) \wedge [-\bar{\mathbf{K}}^i \wedge \mathbf{R}_N + (\mathbf{r} \cdot \bar{\mathbf{K}})^i - 2(-2e^{-2A})\bar{\mathbf{K}}^i \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})], \\
^{(d)}(\mathbf{R}^2)^i_j &= (-2e^{-2A})(-2\partial Adz) \wedge [\mathbf{K}^i \wedge \hat{D}\bar{\mathbf{K}}_j + \hat{D}\bar{\mathbf{K}}^i \wedge \mathbf{K}_j] \\
&\quad + (-2e^{-2A})(-2\bar{\partial}Ad\bar{z}) \wedge [\bar{\mathbf{K}}^i \wedge \hat{D}\mathbf{K}_j + \hat{D}\mathbf{K}^i \wedge \bar{\mathbf{K}}_j], \tag{A.27}
\end{aligned}$$

and

$$\begin{aligned}
^{(0)}(\mathbf{R}^2)^z{}_z &= \mathbf{R}_N^2 + (-2e^{-2A}) \left[2\mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) + (\hat{D}\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) \right] + (-2e^{-2A})^2 (\bar{\mathbf{K}} \cdot \mathbf{K})^2, \\
^{(0)}(\mathbf{R}^2)^{\bar{z}}{}_{\bar{z}} &= \mathbf{R}_N^2 + (-2e^{-2A}) \left[2\mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) + (\hat{D}\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) \right] + (-2e^{-2A})^2 (\bar{\mathbf{K}} \cdot \mathbf{K})^2, \\
^{(0)}(\mathbf{R}^2)^z{}_{\bar{z}} &= (-2e^{-2A})(\hat{D}\bar{\mathbf{K}} \cdot \hat{D}\bar{\mathbf{K}}), \quad ^{(0)}(\mathbf{R}^2)^{\bar{z}}{}_z = (-2e^{-2A})(\hat{D}\mathbf{K} \cdot \hat{D}\mathbf{K}), \\
^{(0)}(\mathbf{R}^2)^z{}_j &= (-2e^{-2A}) \left[\mathbf{R}_N \wedge \hat{D}\bar{\mathbf{K}}_j + (\hat{D}\bar{\mathbf{K}} \cdot \mathbf{r})_j \right] \\
&\quad + (-2e^{-2A})^2 \left[(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge \hat{D}\bar{\mathbf{K}}_j + (\hat{D}\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge \bar{\mathbf{K}}_j + (\hat{D}\bar{\mathbf{K}} \cdot \bar{\mathbf{K}}) \wedge \mathbf{K}_j \right], \\
^{(0)}(\mathbf{R}^2)^{\bar{z}}{}_j &= (-2e^{-2A}) \left[-\mathbf{R}_N \wedge \hat{D}\mathbf{K}_j + (\hat{D}\mathbf{K} \cdot \mathbf{r})_j \right] \\
&\quad + (-2e^{-2A})^2 \left[-(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge \hat{D}\mathbf{K}_j + (\hat{D}\mathbf{K} \cdot \bar{\mathbf{K}}) \wedge \mathbf{K}_j + (\hat{D}\mathbf{K} \cdot \mathbf{K}) \wedge \bar{\mathbf{K}}_j \right], \\
^{(0)}(\mathbf{R}^2)^i{}_z &= \left[\mathbf{R}_N \wedge \hat{D}\mathbf{K}^i + (\mathbf{r} \cdot \hat{D}\mathbf{K})^i \right] \\
&\quad + (-2e^{-2A}) \left[(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge \hat{D}\mathbf{K}^i + \mathbf{K}^i \wedge (\hat{D}\mathbf{K} \cdot \bar{\mathbf{K}}) + \bar{\mathbf{K}}^i \wedge (\hat{D}\mathbf{K} \cdot \mathbf{K}) \right], \\
^{(0)}(\mathbf{R}^2)^i{}_{\bar{z}} &= \left[-\mathbf{R}_N \wedge \hat{D}\bar{\mathbf{K}}^i + (\mathbf{r} \cdot \hat{D}\bar{\mathbf{K}})^i \right] \\
&\quad + (-2e^{-2A}) \left[-(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge \hat{D}\bar{\mathbf{K}}^i + \bar{\mathbf{K}}^i \wedge (\hat{D}\bar{\mathbf{K}} \cdot \mathbf{K}) + \mathbf{K}^i \wedge (\hat{D}\bar{\mathbf{K}} \cdot \bar{\mathbf{K}}) \right],
\end{aligned} \tag{A.28}$$

along with

$$\begin{aligned}
^{(0)}(\mathbf{R}^2)^i{}_j &= (\mathbf{r}^2)^i{}_j + (-2e^{-2A}) \left[\hat{D}\mathbf{K}^i \wedge \hat{D}\bar{\mathbf{K}}_j + \hat{D}\bar{\mathbf{K}}^i \wedge \hat{D}\mathbf{K}_j \right. \\
&\quad \left. + \mathbf{K}^i \wedge (\bar{\mathbf{K}} \cdot \mathbf{r})_j + \bar{\mathbf{K}}^i \wedge (\mathbf{K} \cdot \mathbf{r})_j + (\mathbf{r} \cdot \mathbf{K})^i \wedge \bar{\mathbf{K}}_j + (\mathbf{r} \cdot \bar{\mathbf{K}})^i \wedge \mathbf{K}_j \right] \\
&\quad + (-2e^{-2A})^2 (\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\mathbf{K}^i \wedge \bar{\mathbf{K}}_j - \bar{\mathbf{K}}^i \wedge \mathbf{K}_j).
\end{aligned} \tag{A.29}$$

A.5.7 $\mathbf{I}_{\text{CS}}^{3d}$

By using the above results, we can compute 3d Chern-Simons terms $\mathbf{I}_{\text{CS}}^{3d}$ on the regularized cone geometry. For the contribution of the form $\partial\bar{\partial}Adz \wedge d\bar{z}$ or $\partial A\bar{\partial}Adz \wedge d\bar{z}$ multiplied by zeroth order terms in (z, \bar{z}) -expansion, we obtain

$$\mathbf{I}_{\text{CS}}^{3d}|_{\partial\bar{\partial}A} = (-4)(\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge \Gamma_N, \tag{A.30}$$

$$\mathbf{I}_{\text{CS}}^{3d}|_{\partial A\bar{\partial}A} = 0. \tag{A.31}$$

On the other hand, the contribution of the form ∂A (or $\bar{\partial} A$) times zeroth order terms in (z, \bar{z}) -expansion as well as that of order- ϵ^0 and zeroth order in (z, \bar{z}) -expansion are

$$\begin{aligned} I_{\text{CS}}^{3d}|_{\partial A, \bar{\partial} A} &= (2\partial A dz) \wedge [\mathbf{R}_N + 2(-2e^{-2A})(\bar{\mathbf{K}} \cdot \mathbf{K})] \\ &\quad + (-1)(2\bar{\partial} A d\bar{z}) \wedge [\mathbf{R}_N + 2(-2e^{-2A})(\bar{\mathbf{K}} \cdot \mathbf{K})] , \end{aligned} \quad (\text{A.32})$$

$$I_{\text{CS}}^{3d}|_{0th} = 2\mathbf{\Gamma}_N \wedge \mathbf{R}_N + \text{tr} \left(\gamma \mathbf{r} - \frac{1}{3} \gamma^3 \right) + 2(-2e^{-2A}) [(\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}}) + (\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K})] . \quad (\text{A.33})$$

A.5.8 $\mathcal{P}_{\text{anom}}^{3d}$

From the above expression for \mathbf{R}^2 , we can also straightforwardly compute the anomaly polynomial $\mathcal{P}_{\text{anom}}^{3d} = \text{tr}(\mathbf{R}^2)$ corresponding to the 3d Chern-Simons term. By using the same notation as the 3d Chern-Simons term case just above, we have

$$\mathcal{P}_{\text{anom}}^{3d}|_{\partial \bar{\partial} A} = (-8)(\partial \bar{\partial} A dz \wedge d\bar{z}) \wedge [\mathbf{R}_N + (-2e^{-2A})(\bar{\mathbf{K}} \cdot \mathbf{K})] , \quad (\text{A.34})$$

$$\mathcal{P}_{\text{anom}}^{3d}|_{\partial A \bar{\partial} A} = 16(\partial A \bar{\partial} A dz \wedge d\bar{z}) \wedge (-2e^{-2A})(\bar{\mathbf{K}} \cdot \mathbf{K}) , \quad (\text{A.35})$$

and

$$\begin{aligned} \mathcal{P}_{\text{anom}}^{3d}|_{\partial A, \bar{\partial} A} &= (-8)(\partial A dz) \wedge (-2e^{-2A})(\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}}) \\ &\quad + (-8)(\bar{\partial} A d\bar{z}) \wedge (-2e^{-2A})(\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) , \end{aligned} \quad (\text{A.36})$$

$$\mathcal{P}_{\text{anom}}^{3d}|_{0th} = 4(-2e^{-2A}) \left[\mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) + (\hat{D}\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) - (\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{K}) \right] + 2\mathbf{R}_N^2 + \text{tr}(\mathbf{r}^2) . \quad (\text{A.37})$$

B Details of Computation in Anomaly Polynomial Method

In this Appendix, we will explain in detail the derivation of holographic entanglement entropy formulas for purely gravitational Chern-Simons terms from the anomaly polynomials we have briefly summarized the results in §4.

B.1 $\text{tr}(\mathbf{R}^2)$

Let us first start with the anomaly polynomial $\mathcal{P}_{\text{anom}}^{3d} = \text{tr}(\mathbf{R}^2)$ corresponding to 3d gravitational Chern-Simons term. We will evaluate this polynomial on the regularized cone geometry. As reviewed in §§2.4, there are two types of potential contributions to the holographic entanglement entropy: the contribution proportional to $\partial A \bar{\partial} A$ and that proportional to $\partial \bar{\partial} A$. We will first evaluate them separately and then combine to get the holographic entanglement entropy formula from the anomaly polynomial.

B.1.1 $\partial\bar{\partial}A$ Term

Among the anomaly polynomial $\text{tr}(\mathbf{R}^2)$, there are two possibilities to have the contribution proportional to $\partial\bar{\partial}A$:

$${}^{(d^2)}\mathbf{R}_z^z {}^{(0)}\mathbf{R}_z^z \times 2, \quad {}^{(d^2)}\mathbf{R}_{\bar{z}}^{\bar{z}} {}^{(0)}\mathbf{R}_{\bar{z}}^{\bar{z}} \times 2, \quad (\text{B.1})$$

where the factor 2 comes from the choice for the location of ${}^{(d^2)}\mathbf{R}$ in $\text{tr}(\mathbf{R}^2)$. By using the expression of \mathbf{R} summarized in Eq. (A.9) and summing up the contribution from the above two possibilities, we obtain after integration

$$\begin{aligned} \int_{\text{Cone}_{\Sigma \times I, n}} \mathcal{P}_{\text{anom}}^{3d} |_{\partial\bar{\partial}A} &= \int_{\text{Cone}_{\Sigma \times I, n}} (-8 \partial\bar{\partial}A dz \wedge d\bar{z}) \wedge [\mathbf{R}_N + (-2e^{-2A})(\bar{\mathbf{K}} \cdot \mathbf{K})] \\ &= 8\pi\epsilon \int_{\Sigma \times I} [\mathbf{R}_N - 2(\bar{\mathbf{K}} \cdot \mathbf{K})] . \end{aligned} \quad (\text{B.2})$$

B.1.2 $\partial A \bar{\partial}A$ Term

As for the terms proportional to $\partial A \bar{\partial}A$, there are two possibilities in $\text{tr}(\mathbf{R}^2)$:

$${}^{(d)}\mathbf{R}_z^i {}^{(d)}\mathbf{R}_i^z \times 2, \quad {}^{(d)}\mathbf{R}_{\bar{z}}^i {}^{(d)}\mathbf{R}_i^{\bar{z}} \times 2. \quad (\text{B.3})$$

Here the factor 2 comes from the choice for the location of ${}^{(d)}\mathbf{R}_z^i$ and ${}^{(d)}\mathbf{R}_{\bar{z}}^i$. By summing up and integrating these two contributions, we obtain

$$\begin{aligned} \int_{\text{Cone}_{\Sigma \times I, n}} \mathcal{P}_{\text{anom}}^{3d} |_{\partial A \bar{\partial}A} &= \int_{\text{Cone}_{\Sigma \times I, n}} 16(-2e^{-2A})(\partial A \bar{\partial}A dz \wedge d\bar{z}) \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) \\ &= 16\pi\epsilon \int (\bar{\mathbf{K}} \cdot \mathbf{K}) . \end{aligned} \quad (\text{B.4})$$

B.1.3 Final Result

By summing Eqs. (B.2) and (B.4), we obtain the total of the order ϵ contribution to the anomaly polynomial as follows:

$$8\pi\epsilon \int_{\Sigma \times I} \mathbf{R}_N = 8\pi\epsilon \int_{\Sigma} \mathbf{\Gamma}_N . \quad (\text{B.5})$$

Therefore, the holographic entanglement entropy formula for 3d gravitational Chern-Simons term is computed from the anomaly polynomial as

$$S_{\text{ent,CS}}^{3d} = 8\pi \int_{\Sigma} \mathbf{\Gamma}_N , \quad (\text{B.6})$$

which reproduces the result obtained in [42].

B.2 $\text{tr}(\mathbf{R}^4)$

As a next example, we proceed to 7d and consider the single-trace type of gravitational Chern-Simons term $\mathbf{I}_{\text{CS}}^{7d, \text{single}}$ and the corresponding anomaly polynomial $\mathcal{P}_{\text{anom}}^{7d, \text{single}} = \text{tr}(\mathbf{R}^4)$. In the same ways as 3d case, we first evaluate the terms proportional to $\partial\bar{\partial}A$ and $\partial A \bar{\partial}A$ separately and then combine to get the holographic entanglement entropy formula.

B.2.1 $\partial\bar{\partial}A$ Term

We first evaluate the term proportional to $\partial\bar{\partial}A$ in $\text{tr}(\mathbf{R}^4)$. There are two possibilities in $\text{tr}(\mathbf{R}^4)$:

$$^{(d^2)}\mathbf{R}^z{}_z{}^{(0)}(\mathbf{RRR})^z{}_z \times 4, \quad ^{(d^2)}\mathbf{R}^{\bar{z}}{}_{\bar{z}}{}^{(0)}(\mathbf{RRR})^{\bar{z}}{}_{\bar{z}} \times 4. \quad (\text{B.7})$$

Here $^{(0)}(\mathbf{RRR})^z{}_z = ^{(0)}\mathbf{R}^z{}_\mu{}^{(0)}\mathbf{R}^\mu{}_\nu{}^{(0)}\mathbf{R}^\nu{}_z$ and the factor 4 comes from the choice for the location of $^{(d^2)}\mathbf{R}$ in $\text{tr}(\mathbf{R}^4)$. By summing up these contributions, we obtain after integration

$$\begin{aligned} & \int_{\text{Cone}_{\Sigma \times I, n}} \mathcal{P}_{\text{anom}}^{7d, \text{single}}|_{\partial\bar{\partial}A} \\ &= \int_{\text{Cone}_{\Sigma \times I, n}} (-16\partial\bar{\partial}A dz \wedge d\bar{z}) \\ & \quad \wedge \left[\left[\mathbf{R}_N + (-2e^{-2A})(\bar{\mathbf{K}} \cdot \mathbf{K}) \right]^3 \right. \\ & \quad + 2(-2e^{-2A})[\mathbf{R}_N + (-2e^{-2A})(\bar{\mathbf{K}} \cdot \mathbf{K})] \wedge (\hat{D}\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) + (-2e^{-2A})(\hat{D}\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \hat{D}\mathbf{K}) \\ & \quad \left. + (-2e^{-2A})^2 \left[(\hat{D}\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\hat{D}\mathbf{K} \cdot \bar{\mathbf{K}}) + (\hat{D}\bar{\mathbf{K}} \cdot \bar{\mathbf{K}}) \wedge (\hat{D}\mathbf{K} \cdot \mathbf{K}) \right] \right] \\ &= 16\pi\epsilon \int_{\Sigma \times I} \left[\mathbf{R}_N^3 - 6\mathbf{R}_N^2 \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) - 4\mathbf{R}_N \wedge (\hat{D}\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) + 12\mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 \right. \\ & \quad + 4(\hat{D}\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\hat{D}\mathbf{K} \cdot \bar{\mathbf{K}}) + 4(\hat{D}\bar{\mathbf{K}} \cdot \bar{\mathbf{K}}) \wedge (\hat{D}\mathbf{K} \cdot \mathbf{K}) - 2(\hat{D}\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \hat{D}\mathbf{K}) \\ & \quad \left. + 8(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\hat{D}\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) - 8(\bar{\mathbf{K}} \cdot \mathbf{K})^3 \right]. \quad (\text{B.8}) \end{aligned}$$

B.2.2 $\partial A \bar{\partial} A$ Term

For the terms proportional to $\partial A \bar{\partial} A$, there are eight possibilities in $\text{tr}(\mathbf{R}^4)$:

$$\begin{aligned} & ^{(d)}\mathbf{R}^i{}_z{}^{(d)}\mathbf{R}^z{}_j{}^{(0)}(\mathbf{RR})^j{}_i \times 4, \quad ^{(d)}\mathbf{R}^i{}_{\bar{z}}{}^{(d)}\mathbf{R}^{\bar{z}}{}_j{}^{(0)}(\mathbf{RR})^j{}_i \times 4, \\ & ^{(d)}\mathbf{R}^z{}_j{}^{(d)}\mathbf{R}^j{}_z{}^{(0)}(\mathbf{RR})^z{}_z \times 4, \quad ^{(d)}\mathbf{R}^{\bar{z}}{}_j{}^{(d)}\mathbf{R}^j{}_{\bar{z}}{}^{(0)}(\mathbf{RR})^{\bar{z}}{}_{\bar{z}} \times 4, \\ & ^{(d)}\mathbf{R}^i{}_z{}^{(0)}\mathbf{R}^z{}_z{}^{(d)}\mathbf{R}^z{}_j{}^{(0)}\mathbf{R}^j{}_i \times 4, \quad ^{(d)}\mathbf{R}^i{}_{\bar{z}}{}^{(0)}\mathbf{R}^{\bar{z}}{}_{\bar{z}}{}^{(d)}\mathbf{R}^{\bar{z}}{}_j{}^{(0)}\mathbf{R}^j{}_i \times 4, \\ & ^{(d)}\mathbf{R}^i{}_z{}^{(0)}\mathbf{R}^z{}_j{}^{(d)}\mathbf{R}^j{}_{\bar{z}}{}^{(0)}\mathbf{R}^{\bar{z}}{}_i \times 4, \quad ^{(d)}\mathbf{R}^z{}_j{}^{(0)}\mathbf{R}^j{}_{\bar{z}}{}^{(d)}\mathbf{R}^{\bar{z}}{}_i{}^{(0)}\mathbf{R}^i{}_z \times 4. \end{aligned} \quad (\text{B.9})$$

Here the factor 4 comes from the locations of $^{(d)}\mathbf{R}$'s in $\text{tr}(\mathbf{R}^4)$. Each term in this list is then computed as follows :

$$\begin{aligned}
& ^{(d)}\mathbf{R}_z^i {}^{(d)}\mathbf{R}_j^z {}^{(0)}(\mathbf{R}\mathbf{R})^j{}_i \times 4 \\
& = 16(-2e^{-2A})(\partial A \bar{\partial} A dz \wedge d\bar{z}) \\
& \quad \wedge \left[-(-2e^{-2A})(\hat{D}\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\hat{D}\mathbf{K} \cdot \bar{\mathbf{K}}) + (\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{r} \cdot \mathbf{K}) + (-2e^{-2A})(\hat{D}\bar{\mathbf{K}} \cdot \bar{\mathbf{K}}) \wedge (\hat{D}\mathbf{K} \cdot \mathbf{K}) \right. \\
& \quad \left. + 2(-2e^{-2A})(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{K}) + (-2e^{-2A})^2(\bar{\mathbf{K}} \cdot \mathbf{K})^3 \right], \\
& ^{(d)}\mathbf{R}_z^i {}^{(d)}\mathbf{R}_j^z {}^{(0)}(\mathbf{R}\mathbf{R})^j{}_i \times 4 = ^{(d)}\mathbf{R}_{\bar{z}}^i {}^{(d)}\mathbf{R}_{\bar{j}}^z {}^{(0)}(\mathbf{R}\mathbf{R})^j{}_i \times 4, \\
& ^{(d)}\mathbf{R}_j^z {}^{(d)}\mathbf{R}_z^j {}^{(0)}(\mathbf{R}\mathbf{R})^z{}_z \times 4 \\
& = 16(-2e^{-2A})(\partial A \bar{\partial} A dz \wedge d\bar{z}) \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) \\
& \quad \wedge \left[[\mathbf{R}_N + (-2e^{-2A})(\bar{\mathbf{K}} \cdot \mathbf{K})]^2 + (-2e^{-2A})(\hat{D}\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) \right], \\
& ^{(d)}\mathbf{R}_{\bar{j}}^z {}^{(d)}\mathbf{R}_{\bar{z}}^j {}^{(0)}(\mathbf{R}\mathbf{R})^{\bar{z}}{}_{\bar{z}} \times 4 = ^{(d)}\mathbf{R}_j^z {}^{(d)}\mathbf{R}_z^j {}^{(0)}(\mathbf{R}\mathbf{R})^z{}_z \times 4, \\
& ^{(d)}\mathbf{R}_z^i {}^{(0)}\mathbf{R}_z^z {}^{(d)}\mathbf{R}_j^z {}^{(0)}\mathbf{R}_i^j \times 4 = 16(-2e^{-2A})(\partial A \bar{\partial} A dz \wedge d\bar{z}) \wedge [\mathbf{R}_N + (-2e^{-2A})(\bar{\mathbf{K}} \cdot \mathbf{K})] \\
& \quad \wedge [(\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{K}) + (-2e^{-2A})(\bar{\mathbf{K}} \cdot \mathbf{K})^2], \\
& ^{(d)}\mathbf{R}_{\bar{z}}^i {}^{(0)}\mathbf{R}_{\bar{z}}^z {}^{(d)}\mathbf{R}_{\bar{j}}^z {}^{(0)}\mathbf{R}_i^j \times 4 = ^{(d)}\mathbf{R}_z^i {}^{(0)}\mathbf{R}_z^z {}^{(d)}\mathbf{R}_j^z {}^{(0)}\mathbf{R}_i^j \times 4, \\
& ^{(d)}\mathbf{R}_z^i {}^{(0)}\mathbf{R}_j^z {}^{(d)}\mathbf{R}_{\bar{z}}^j {}^{(0)}\mathbf{R}_{\bar{i}}^z \times 4 = 16(-2e^{-2A})^2(\partial A \bar{\partial} A dz \wedge d\bar{z}) \wedge (\hat{D}\bar{\mathbf{K}} \cdot \bar{\mathbf{K}}) \wedge (\hat{D}\mathbf{K} \cdot \mathbf{K}), \\
& ^{(d)}\mathbf{R}_j^z {}^{(0)}\mathbf{R}_{\bar{z}}^j {}^{(d)}\mathbf{R}_{\bar{i}}^z {}^{(0)}\mathbf{R}_z^i \times 4 = 16(-2e^{-2A})^2(\partial A \bar{\partial} A dz \wedge d\bar{z}) \wedge (\hat{D}\bar{\mathbf{K}} \cdot \bar{\mathbf{K}}) \wedge (\hat{D}\mathbf{K} \cdot \mathbf{K}).
\end{aligned} \tag{B.10}$$

By summing them up and doing integration, we obtain

$$\begin{aligned}
& \int_{\text{Cone}_{\Sigma \rtimes I, n}} \mathcal{P}_{\text{anom}}^{7d, \text{single}}|_{\partial A \bar{\partial} A} \\
&= \int_{\text{Cone}_{\Sigma \rtimes I, n}} 32(-2e^{-2A})(\partial A \bar{\partial} A dz \wedge d\bar{z}) \\
& \quad \wedge \left[\mathbf{R}_N^2 \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) + \mathbf{R}_N \wedge [3(-2e^{-2A})(\bar{\mathbf{K}} \cdot \mathbf{K})^2 + (\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{K})] \right. \\
& \quad + 2(-2e^{-2A})(\hat{D}\bar{\mathbf{K}} \cdot \bar{\mathbf{K}}) \wedge (\hat{D}\mathbf{K} \cdot \mathbf{K}) - (-2e^{-2A})(\hat{D}\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\hat{D}\mathbf{K} \cdot \bar{\mathbf{K}}) + (\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{r} \cdot \mathbf{K}) \\
& \quad + (-2e^{-2A})(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\hat{D}\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) + 3(-2e^{-2A})(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{K}) \\
& \quad \left. - 3(-2e^{-2A})^2(\bar{\mathbf{K}} \cdot \mathbf{K})^3 \right] \\
&= 16\pi\epsilon \int_{\Sigma \rtimes I} \left[2\mathbf{R}_N^2 \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) + \mathbf{R}_N \wedge [-6(\bar{\mathbf{K}} \cdot \mathbf{K})^2 + 2(\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{K})] \right. \\
& \quad + 2(\hat{D}\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\hat{D}\mathbf{K} \cdot \bar{\mathbf{K}}) - 4(\hat{D}\bar{\mathbf{K}} \cdot \bar{\mathbf{K}}) \wedge (\hat{D}\mathbf{K} \cdot \mathbf{K}) + 2(\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{r} \cdot \mathbf{K}) \\
& \quad \left. - 2(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\hat{D}\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) - 6(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{K}) + 8(\bar{\mathbf{K}} \cdot \mathbf{K})^3 \right]. \quad (\text{B.12})
\end{aligned}$$

B.2.3 Final Result

Summing these two contributions in Eqs. (B.8) and (B.12), we obtain

$$\begin{aligned}
16\pi\epsilon \int_{\Sigma \rtimes I} & \left[\mathbf{R}_N^3 - 4\mathbf{R}_N^2 \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) + \mathbf{R}_N \wedge [-4(\hat{D}\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) + 6(\bar{\mathbf{K}} \cdot \mathbf{K})^2 + 2(\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{K})] \right. \\
& + 6(\hat{D}\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\hat{D}\mathbf{K} \cdot \bar{\mathbf{K}}) + 6(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\hat{D}\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) - 6(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{K}) \\
& \left. - 2(\hat{D}\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \hat{D}\mathbf{K}) + 2(\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{r} \cdot \mathbf{K}) \right]. \quad (\text{B.13})
\end{aligned}$$

Now we rewrite this into the form $d(\dots)$. For this purpose, we note the following identities:

$$\begin{aligned}
& d[\mathbf{\Gamma}_N \wedge \mathbf{R}_N^2] = \mathbf{R}_N^3 \text{ along with} \\
& d \left[(-2\mathbf{R}_N) \wedge [(\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}}) + (\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K})] \right] \\
& \quad = 4 \left[\mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{K}) - \mathbf{R}_N^2 \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) - \mathbf{R}_N \wedge (\hat{D}\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) \right], \\
& d \left[3(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge [(\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}}) + (\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K})] \right] \\
& \quad = 6 \left[\mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 + (\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\hat{D}\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) - (\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{K}) - (\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) \wedge (\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}}) \right], \\
& d \left[(\hat{D}\mathbf{K} \cdot \mathbf{r} \cdot \bar{\mathbf{K}}) - (\hat{D}\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{K}) \right] \\
& \quad = 2 \left[(\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{r} \cdot \mathbf{K}) - \mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{K}) - (\hat{D}\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \hat{D}\mathbf{K}) \right], \quad (\text{B.14})
\end{aligned}$$

where we have used

$$\hat{D}^2 \mathbf{K}^i = (\mathbf{r} \cdot \mathbf{K})^i - \mathbf{R}_N \wedge \mathbf{K}^i, \quad \hat{D}^2 \bar{\mathbf{K}}^i = (\mathbf{r} \cdot \bar{\mathbf{K}})^i + \mathbf{R}_N \wedge \bar{\mathbf{K}}^i. \quad (\text{B.15})$$

By using the above identities, we can rewrite Eq. (B.13) as

$$16\pi\epsilon \int_{\Sigma \times I} d \left[\mathbf{\Gamma}_N \wedge \mathbf{R}_N^2 + [-2\mathbf{R}_N + 3(\bar{\mathbf{K}} \cdot \mathbf{K})] \wedge [(\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}}) + (\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K})] \right. \\ \left. + (\hat{D}\mathbf{K} \cdot \mathbf{r} \cdot \bar{\mathbf{K}}) - (\hat{D}\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{K}) \right]. \quad (\text{B.16})$$

Therefore, the holographic entanglement entropy formula for the 7d single-trace gravitational Chern-Simons term is obtained from the anomaly polynomial as

$$S_{\text{ent,CS}}^{7d, \text{single}} \\ = 2(2\pi)(4) \int_{\Sigma} \left[\mathbf{\Gamma}_N \wedge \mathbf{R}_N^2 + [-2\mathbf{R}_N + 3(\bar{\mathbf{K}} \cdot \mathbf{K})] \wedge [(\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}}) + (\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K})] \right. \\ \left. + (\hat{D}\mathbf{K} \cdot \mathbf{r} \cdot \bar{\mathbf{K}}) - (\hat{D}\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{K}) \right]. \quad (\text{B.17})$$

B.3 $\text{tr}(\mathbf{R}^2) \wedge \text{tr}(\mathbf{R}^2)$

As a final example, here we derive the holographic entanglement entropy formula for 7d double-trace gravitational Chern-Simons terms by evaluating the anomaly polynomial $\mathcal{P}_{\text{anom}}^{7d, \text{double}} = \text{tr}(\mathbf{R}^2) \wedge \text{tr}(\mathbf{R}^2)$ on the regularized cone background.

B.3.1 $\partial\bar{\partial}A$ Term

There are two possibilities for this type of contribution in $\text{tr}(\mathbf{R}^2) \wedge \text{tr}(\mathbf{R}^2)$:

$$^{(d^2)}\mathbf{R}^z{}_z{}^{(0)}\mathbf{R}^z{}_z \wedge \mathcal{P}_{\text{anom}}^{3d}|_{0th} \times 4, \quad ^{(d^2)}\mathbf{R}^{\bar{z}}{}_{\bar{z}}{}^{(0)}\mathbf{R}^{\bar{z}}{}_{\bar{z}} \wedge \mathcal{P}_{\text{anom}}^{3d}|_{0th} \times 4, \quad (\text{B.18})$$

where the factor 4 comes from the choice for the location of $^{(d^2)}\mathbf{R}$. By summing over these contributions and doing integration, we obtain

$$\int_{\text{Cone}_{\Sigma \times I, n}} \mathcal{P}_{\text{anom}}^{7d, \text{double}}|_{\partial\bar{\partial}A} \\ = \int_{\text{Cone}_{\Sigma \times I, n}} (-16 \partial\bar{\partial}A dz \wedge d\bar{z}) \wedge [\mathbf{R}_N + (-2e^{-2A})(\bar{\mathbf{K}} \cdot \mathbf{K})] \wedge \mathcal{P}_{\text{anom}}^{3d}|_{0th} \\ = 16\pi\epsilon [\mathbf{R}_N + (-2)(\bar{\mathbf{K}} \cdot \mathbf{K})] \\ \wedge \left[2\mathbf{R}_N^2 - 8\mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) - 8(\hat{D}\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) + \text{tr}(\mathbf{r}^2) + 8(\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{K}) \right]. \quad (\text{B.19})$$

Here we have used the expression of $\mathcal{P}_{\text{anom}}^{3d}|_{0th}$ given in Eq. (A.37).

B.3.2 $\partial A \bar{\partial} A$ Term

As a next step, we evaluate the terms proportional to $\partial A \bar{\partial} A$. There are four possibilities in $\text{tr}(\mathbf{R}^2) \wedge \text{tr}(\mathbf{R}^2)$:

$$\begin{aligned} & {}^{(d)}\mathbf{R}_z^i {}^{(d)}\mathbf{R}_i^z \wedge \mathcal{P}_{\text{anom}}^{3d}|_{0th} \times 4, \quad {}^{(d)}\mathbf{R}_{\bar{z}}^i {}^{(d)}\mathbf{R}_i^{\bar{z}} \wedge \mathcal{P}_{\text{anom}}^{3d}|_{0th} \times 4, \\ & {}^{(d)}\mathbf{R}_z^i {}^{(0)}\mathbf{R}_i^z \wedge {}^{(d)}\mathbf{R}_j^z {}^{(0)}\mathbf{R}_z^j \times 8, \quad {}^{(d)}\mathbf{R}_{\bar{z}}^i {}^{(0)}\mathbf{R}_i^{\bar{z}} \wedge {}^{(d)}\mathbf{R}_{\bar{j}}^{\bar{z}} {}^{(0)}\mathbf{R}_{\bar{z}}^j \times 8, \\ & {}^{(d)}\mathbf{R}_z^i {}^{(0)}\mathbf{R}_i^z \wedge {}^{(d)}\mathbf{R}_{\bar{j}}^j {}^{(0)}\mathbf{R}_{\bar{z}}^{\bar{j}} \times 8, \quad {}^{(d)}\mathbf{R}_i^z {}^{(0)}\mathbf{R}_z^i \wedge {}^{(d)}\mathbf{R}_{\bar{j}}^{\bar{z}} {}^{(0)}\mathbf{R}_{\bar{z}}^j \times 8, \end{aligned} \quad (\text{B.20})$$

and each term is evaluated as follows :

$$\begin{aligned} & {}^{(d)}\mathbf{R}_z^i {}^{(d)}\mathbf{R}_i^z \wedge \mathcal{P}_{\text{anom}}^{3d}|_{0th} \times 4 = 16(-2e^{-2A})(\partial A \bar{\partial} A dz \wedge d\bar{z}) \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge \mathcal{P}_{\text{anom}}^{3d}|_{0th}, \\ & {}^{(d)}\mathbf{R}_{\bar{z}}^i {}^{(d)}\mathbf{R}_i^{\bar{z}} \wedge \mathcal{P}_{\text{anom}}^{3d}|_{0th} \times 4 = {}^{(d)}\mathbf{R}_z^i {}^{(d)}\mathbf{R}_i^z \wedge \mathcal{P}_{\text{anom}}^{3d}|_{0th} \times 4, \\ & {}^{(d)}\mathbf{R}_z^i {}^{(0)}\mathbf{R}_i^z \wedge {}^{(d)}\mathbf{R}_j^z {}^{(0)}\mathbf{R}_z^j \times 8 \\ & \quad = -32(-2e^{-2A})^2(\partial A \bar{\partial} A dz \wedge d\bar{z}) \wedge (\hat{D}\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\hat{D}\mathbf{K} \cdot \bar{\mathbf{K}}), \\ & {}^{(d)}\mathbf{R}_{\bar{z}}^i {}^{(0)}\mathbf{R}_i^{\bar{z}} \wedge {}^{(d)}\mathbf{R}_{\bar{j}}^{\bar{z}} {}^{(0)}\mathbf{R}_{\bar{z}}^j \times 8 = {}^{(d)}\mathbf{R}_z^i {}^{(0)}\mathbf{R}_i^z \wedge {}^{(d)}\mathbf{R}_j^z {}^{(0)}\mathbf{R}_z^j \times 8 \\ & \quad = {}^{(d)}\mathbf{R}_z^i {}^{(0)}\mathbf{R}_i^z \wedge {}^{(d)}\mathbf{R}_{\bar{j}}^j {}^{(0)}\mathbf{R}_{\bar{z}}^{\bar{j}} \times 8 = {}^{(d)}\mathbf{R}_i^z {}^{(0)}\mathbf{R}_z^i \wedge {}^{(d)}\mathbf{R}_{\bar{j}}^{\bar{z}} {}^{(0)}\mathbf{R}_{\bar{z}}^j \times 8. \end{aligned} \quad (\text{B.21})$$

By summing up these terms and doing integration, we obtain

$$\begin{aligned} & \int_{\text{Cone}_{\Sigma \times I, n}} \mathcal{P}_{\text{anom}}^{7d, \text{double}}|_{\partial A \bar{\partial} A} \\ & = \int_{\text{Cone}_{\Sigma \times I, n}} 32(-2e^{-2A})(\partial A \bar{\partial} A dz \wedge d\bar{z}) \\ & \quad \wedge \left[(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge \mathcal{P}_{\text{anom}}^{3d}|_{0th} - 4(-2e^{-2A})(\hat{D}\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\hat{D}\mathbf{K} \cdot \bar{\mathbf{K}}) \right] \\ & = 32\pi\epsilon \int_{\Sigma \times I} \left[2\mathbf{R}_N^2 \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) - 4\mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 - 4(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\hat{D}\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) \right. \\ & \quad \left. + 4(\hat{D}\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\hat{D}\mathbf{K} \cdot \bar{\mathbf{K}}) + 4(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{K}) + (\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge \text{tr}(\mathbf{r}^2) \right]. \end{aligned} \quad (\text{B.22})$$

B.3.3 Final Result

By summing the above two contributions (B.19) and (B.22) and rewriting in the total derivative form, we obtain

$$32\pi\epsilon \int_{\Sigma \times I} d \left[\mathbf{\Gamma}_N \wedge \mathbf{R}_N^2 + \frac{1}{2} \mathbf{\Gamma}_N \wedge \text{tr}(\mathbf{r}^2) + 2 \left[-\mathbf{R}_N + (\bar{\mathbf{K}} \cdot \mathbf{K}) \right] \wedge [(\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}}) + (\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K})] \right]. \quad (\text{B.23})$$

Therefore, the holographic entanglement entropy formula for 7d double-trace gravitational Chern-Simons term obtained from the anomaly polynomial is

$$\begin{aligned}
S_{\text{ent,CS}}^{7d, \text{double}} &= 2^2(2\pi)(4) \int_{\Sigma} \left[\Gamma_N \wedge \mathbf{R}_N^2 + \frac{1}{2} \Gamma_N \wedge \text{tr}(\mathbf{r}^2) \right. \\
&\quad \left. + 2 \left[-\mathbf{R}_N + (\bar{\mathbf{K}} \cdot \mathbf{K}) \right] \wedge [(\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}}) + (\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K})] \right].
\end{aligned}
\tag{B.24}$$

C Alternative Derivation from Anomaly Polynomial: Application of Dong's Formula

This Appendix provides an alternative way to derive the holographic entanglement entropy formulas from the anomaly polynomials. The idea is that, since the anomaly polynomials depend only on the Riemann tensor, we directly apply Dong's formula [8] to the anomaly polynomials and rewrite the integrand in the total derivative form, instead of doing ϵ -expansion near the regularized cone geometry explicitly. We will confirm that the results in §3 are reproduced correctly.

Here is one remark: Since we do not carry out ϵ -expansion near the regularized cone geometry explicitly in this Appendix, we will take the coordinate to be general (for example, we do not introduce (z, \bar{z}) coordinate), except when we compare the final results with those obtained by using ϵ -expansion in the rest part of this paper. For the comparison, we summarize in Appendix C.6 some useful formulas that convert the notation in this Appendix into the one we used in the main text of this paper.

C.1 Some Geometry of Co-Dimension Two Surface in Differential Form

Before introducing the Dong's holographic entanglement entropy formula and applying it to the anomaly polynomials, we start with a brief summary of geometry relevant to a co-dimension two surface $\tilde{\Sigma}$ living in a spacetime (in Euclidean signature) with metric $G_{\mu\nu}$.

For the co-dimension two surface $\tilde{\Sigma}$, we can introduce two normal vectors $n_{(a)}^\mu$ (where $a = 1, 2$) such that

$$n_{(1)}^\mu n_{(1)\mu} = n_{(2)}^\mu n_{(2)\mu} = 1, \quad n_{(1)}^\mu n_{(2)\mu} = 0. \tag{C.1}$$

Then the induced metric on the tangent space spanned by these normal vectors is given by

$$n_{ab} = n_{(a)}^\mu n_{(b)}^\nu G_{\mu\nu}. \tag{C.2}$$

We also denote the binormal on $\tilde{\Sigma}$ by ϵ_{ab} . Here we note that indices a, b, c, \dots denote the directions orthogonal to $\tilde{\Sigma}$, while we labels the ones along $\tilde{\Sigma}$ with i, j, k, l, \dots (we use y^i for the coordinates along $\tilde{\Sigma}$).²¹

By using these quantities, the extrinsic curvatures on the co-dimension two surface $\tilde{\Sigma}$ are given by

$$K_{aij} \equiv D_i n_{aj}|_{\tilde{\Sigma}} = \frac{1}{2} \partial_a G_{ij}, \quad K_{aij} = K_{aji}, \quad (\text{C.3})$$

where D_i is the covariant derivative compatible with the induced metric on $\tilde{\Sigma}$. It is also useful for our purpose to define “gauge field” $(\Gamma_N)_i$ for the $SO(2)$ normal bundle rotation and its field strength $(R_N)_{ij}$ as

$$(\Gamma_N)_i = \frac{1}{2} i \epsilon^{ab} \partial_a G_{ib}, \quad (R_N)_{ij} \equiv 2 \partial_{[i} (\Gamma_N)_{j]}. \quad (\text{C.4})$$

Then the Christoffel symbol and Riemann tensor are written as

$$\Gamma_{bc}^\mu = 0, \quad \Gamma_{bj}^a = i \epsilon^a_b (\Gamma_N)_j, \quad \Gamma_{ij}^a = -K_{ij}^a, \quad \Gamma_{bj}^i = K_{bj}^i, \quad (\text{C.5})$$

and

$$\begin{aligned} R_{abij} &= i \epsilon_{ab} (R_N)_{ij} + K_{bil} K_a^l{}_j - K_{ail} K_b^l{}_j, \\ R_{ijkl} &= r_{ijkl} - K_{aik} K^a{}_{jl} + K_{ail} K^a{}_{jk}, \\ R_{ajkl} &= -D_k K_{ajl} + D_l K_{ajk} - 2i \epsilon_{ab} (\Gamma_N)_{[k} K^b{}_{l]j}. \end{aligned} \quad (\text{C.6})$$

Here r_{ijkl} is the Riemann tensor for the induced metric along $\tilde{\Sigma}$.

For our purpose to deal with the anomaly polynomials, it is convenient to introduce the differential forms for some geometric quantities defined above:

$$\begin{aligned} \Gamma_{\nu}^\mu &\equiv \Gamma_{\nu k}^\mu dy^k, \quad \Gamma_N \equiv (\Gamma_N)_i dy^i, \quad R_N \equiv d\Gamma_N, \quad K_{ai} \equiv K_{aik} dy^k, \\ R_{\mu\nu} &\equiv \frac{1}{2} R_{\mu\nu kl} dy^k \wedge dy^l, \quad r_{ij} \equiv \frac{1}{2} r_{ijkl} dy^k \wedge dy^l. \end{aligned} \quad (\text{C.7})$$

More explicitly, by substituting the above component-based expressions to these definitions, we obtain

$$\Gamma_{b}^a = i \epsilon^a_b \Gamma_N, \quad \Gamma_i^a = -K_{ij}^a, \quad \Gamma_b^i = K_{bj}^i, \quad (\text{C.8})$$

$$\begin{aligned} R_{b}^a &= i \epsilon^a_b [R_N + \sigma], \quad R_{j}^i = r_{ij}^i - 2w_{j}^i, \\ R_i^a &= -\hat{D} K_{ij}^a, \quad \hat{D} K_{aj} \equiv D K_{aj} + i \epsilon_{ab} \Gamma_N \wedge K_{j}^b, \end{aligned} \quad (\text{C.9})$$

²¹For the geometry of the co-dimension two surface, we follow the notation and discussion in the Appendix B of [57]. Here, we only summarize some quantities that we need to build the differential forms on $\tilde{\Sigma}$ used for the derivation of the holographic entanglement entropy formula (for example, for the Riemann tensor, we only summarize $R_{\mu\nu ij}$ components only). We also note that we chose coordinates as $n_{(a)}^\mu = \delta_a^\mu$ such that the index ‘ a ’ also means a coordinate. One could equally think of all the a, b, \dots indices as projection on the normal vectors.

where \hat{D} is the covariant derivative associated with the normal bundle gauge field $\mathbf{\Gamma}_N$. We have also defined

$$\begin{aligned}\sigma_{ij} &\equiv \frac{1}{2}\epsilon^{ab}K_{ai}\wedge K_{bj}, \quad \sigma_{ij} = \sigma_{ji}, \quad K_{al}\wedge K_b{}^l = (-i\epsilon_{ab})\sigma, \quad \sigma \equiv \sigma^i{}_i, \\ \mathbf{w}_{ij} &\equiv \frac{1}{2}K^a{}_i\wedge K_{aj}, \quad \mathbf{w}_{ij} = -\mathbf{w}_{ji}.\end{aligned}\tag{C.10}$$

To show the third relation in the first line, we have used $\epsilon^{ab}\epsilon_{ab} = 2$ which follows from $\epsilon_{ab}\epsilon_{cd} = n_{ac}n_{bd} - n_{ad}n_{bc}$ and $n^a{}_a = 2$. For later use, here we summarize some useful identities relevant to \mathbf{w}_{ij} and σ_{ij} :²²

$$\begin{aligned}\mathbf{w}_{ik}\wedge\mathbf{w}^k{}_j &= -\frac{1}{2}\sigma\wedge\sigma_{ij}, \quad \text{tr}[\omega^2] = -\frac{1}{2}\sigma^2, \quad \sigma_{ik}\wedge\sigma^k{}_j = -\frac{1}{2}\sigma\wedge\sigma_{ij}, \quad \text{tr}[\sigma^2] = -\frac{1}{2}\sigma^2, \\ \mathbf{w}^i{}_k\wedge\sigma^k{}_j &= -\frac{1}{2}\sigma\wedge\mathbf{w}^i{}_j, \quad \sigma^i{}_k\wedge\mathbf{w}^k{}_j = -\frac{1}{2}\sigma\wedge\mathbf{w}^i{}_j.\end{aligned}\tag{C.11}$$

C.2 Dong's Formula for Holographic Entanglement Entropy

In [8], the holographic entanglement entropy formula is derived for a general theory with a Lagrangian which is a functional of Riemann tensor and does not involve covariant derivatives of Riemann tensor. The purpose of this part of Appendix is to rewrite this formula in a way it can be readily applied to the anomaly polynomials. Here we focus only on pure gravitational anomaly polynomials since the extension to mixed anomaly polynomials is straightforward.

In Dong's formula, there is a parameter q_α which essentially counts the number of the regularization factor e^{-2A} carried by each term (labeled by the subscript α) in the second order derivative of the Lagrangian with respect to Riemann tensor evaluated on the regularized cone background. For our purposes, Q 's and V 's in [8] do not contribute to the final answer. Using this, we can simplify the Dong's rule to compute q_α to the following: (1) first expand the product of \mathbf{R} 's into all lower-indices, (2) then keep track of q_α , which counts half of the total number of \mathbf{K}_{ai} in each term in the sum. To implement this, it is convenient to introduce \mathbf{R}_t as

$$(\mathbf{R}_t)^a{}_b \equiv i\epsilon^a{}_b(\mathbf{R}_N + t\sigma), \quad (\mathbf{R}_t)^i{}_j \equiv \mathbf{r}^i{}_j - t(2\mathbf{w}^i{}_j), \quad (\mathbf{R}_t)^a{}_j \equiv -t^{1/2}\hat{D}\mathbf{K}^a{}_j. \tag{C.12}$$

We note that σ_{ij} and ω_{ij} each gets a factor t since they are quadratic in \mathbf{K}_{ai} . By noticing that

$$\int_0^1 dt \, t^{q_\alpha} = \frac{1}{1+q_\alpha}, \tag{C.13}$$

²²It should be clear that whenever σ appears inside a $\text{tr}[\dots]$, it is $\sigma^i{}_j$. Otherwise $\sigma = \sigma^i{}_i$.

we can now write a covariant form of Dong's formula derived in [8] specialised to our case as:

$$\begin{aligned}
S_{\text{ent}} &= S_{\text{ent}}^{\text{Wald}} + S_{\text{ent}}^{KK} \quad \text{where} \\
S_{\text{ent}}^{\text{Wald}} &\equiv 2\pi \int_{\tilde{\Sigma}} \left[-\frac{\partial L}{\partial R_{\mu\rho\nu\sigma}} \epsilon_{\mu\rho} \epsilon_{\nu\sigma} \right] (\tilde{\Sigma}^* 1), \\
S_{\text{ent}}^{KK} &\equiv 4\pi \int_{\tilde{\Sigma}} \int_0^1 dt \left\{ \left(\frac{\partial^2 L}{\partial R_{\mu_1\rho_1\nu_1\sigma_1} \partial R_{\mu_2\rho_2\nu_2\sigma_2}} \right)_t K_{l_1\rho_1\sigma_1} K_{l_2\rho_2\sigma_2} \right. \\
&\quad \left. \times \left[(n_{\mu_1\mu_2} n_{\nu_1\nu_2} - \epsilon_{\mu_1\mu_2} \epsilon_{\nu_1\nu_2}) n^{l_1 l_2} + (n_{\mu_1\mu_2} \epsilon_{\nu_1\nu_2} + \epsilon_{\mu_1\mu_2} n_{\nu_1\nu_2}) \epsilon^{l_1 l_2} \right] \right\} (\tilde{\Sigma}^* 1).
\end{aligned} \tag{C.14}$$

Here the quantities with subscript t means that the curvature two-forms are evaluated by using Eq. (C.12) and $\tilde{\Sigma}^*(\dots)$ denotes the Hodge dual of (\dots) on $\tilde{\Sigma}$.

In the rest of this Appendix, we use the above expression to reproduce the holographic entanglement entropy formula for 3d and 7d gravitational Chern-Simons terms from the anomaly polynomial à la Dong's formula. For the co-dimension two surface $\tilde{\Sigma}$, since we are using the anomaly polynomial after uplifting to a one-dimension higher spacetime as explained in §2, we take as $\tilde{\Sigma} = \Sigma \rtimes I$ where Σ is the bulk entangling surface for the Chern-Simons terms and $I = [0, \infty)$ is a half line. For the reader's convenience, in the end of this Appendix (see Appendix C.6), we have also summarized some useful relations for converting the notation in this Appendix into the one we used in the computation based on ϵ -expansion.

C.3 $\mathcal{P}_{\text{anom}} = \text{tr}(\mathbf{R}^2)$

As the simplest example, let us start with the anomaly polynomial corresponding to 3d gravitational Chern-Simons term and apply Dong's formula. The corresponding Euclidean Lagrangian L is given by:²³

$$L = \frac{1}{(2!)^2} R^\mu{}_{\rho\alpha_1\alpha_2} R^\rho{}_{\mu\alpha_3\alpha_4} (-i\epsilon^{\alpha_1\cdots\alpha_4}). \tag{C.15}$$

Our convention for the epsilon-tensor is that $\epsilon^{ab\alpha_1\alpha_2} = \epsilon^{ab} \epsilon^{\alpha_1\alpha_2}$ where ϵ^{ab} in components are given in Appendix C.6. By using $i\epsilon^a{}_b \mathbf{R}^b{}_a = 2(\mathbf{R}_N + \boldsymbol{\sigma})$, the Wald term $S_{\text{ent}}^{\text{Wald}}$ is given by

$$S_{\text{ent}}^{\text{Wald}} = 8\pi \int_{\Sigma \rtimes I} (\mathbf{R}_N + \boldsymbol{\sigma}), \tag{C.16}$$

while S_{ent}^{KK} term is evaluated as

$$S_{\text{ent}}^{KK} = -8\pi \int_{\Sigma \rtimes I} \boldsymbol{\sigma}. \tag{C.17}$$

Combining these two contributions, we obtain

$$S_{\text{ent,CS}}^{3d} = 8\pi \int_{\Sigma \rtimes I} \mathbf{R}_N = 8\pi \int_{\Sigma \rtimes I} d\mathbf{\Gamma}_N = 8\pi \int_{\Sigma} \mathbf{\Gamma}_N. \tag{C.18}$$

²³We note that our notation for the epsilon tensors deviates from [39] by $(-i)$: $\epsilon_{\text{JLY}}^{ab} = (-i)\epsilon_{\text{here}}^{ab}$ and $\epsilon_{\text{JLY}}^{\alpha_1\alpha_2\alpha_3\alpha_4} = (-i)\epsilon_{\text{here}}^{\alpha_1\alpha_2\alpha_3\alpha_4}$.

This indeed reproduces the result in Eq. (1.2) we have obtained by applying ϵ -expansion directly.

C.4 $\mathcal{P}_{\text{anom}} = \text{tr}(\mathbf{R}^4)$

As a next example, here we consider the anomaly polynomial corresponding to the 7d single-trace gravitational Chern-Simons term. First of all, the Wald term $S_{\text{ent}}^{\text{Wald}}$ is obtained as

$$\begin{aligned} S_{\text{ent}}^{\text{Wald}} &= 2\pi \times 4 \int_{\Sigma \times I} \left\{ \mathbf{R}_N \wedge (\text{tr}(\mathbf{R}^2) - \text{tr}[\mathbf{r}^2]) + 2\mathbf{R}_N^2 \wedge \boldsymbol{\sigma} + 2\mathbf{R}_N \wedge \text{tr}[\mathbf{r}\mathbf{w}] + 2\text{tr}[\mathbf{r}^2\boldsymbol{\sigma}] + 2\mathbf{R}_N \wedge \boldsymbol{\sigma}^2 \right. \\ &\quad + 4\boldsymbol{\sigma} \wedge \text{tr}[\mathbf{r}\mathbf{w}] + 2(i\epsilon^a{}_b)(\hat{D}\mathbf{K}^b) \cdot \boldsymbol{\omega} \cdot (\hat{D}\mathbf{K}_a) - 2\boldsymbol{\sigma} \wedge d[\mathbf{K}^a \cdot (\hat{D}\mathbf{K}_a)] + 2\boldsymbol{\sigma}^3 \\ &\quad \left. + d[(-i\epsilon_{ab})(\mathbf{K}^b \cdot \mathbf{r} \cdot \hat{D}\mathbf{K}^a)] \right\}, \end{aligned} \quad (\text{C.19})$$

where the ‘ \cdot ’ indicates contraction of the i -type indices.

As a next step, let us compute S_{ent}^{KK} term. Since the integrand is computed as

$$\begin{aligned} &4\pi \left(\frac{\partial^2 L}{\partial R_{\mu_1 \rho_1 \nu_1 \sigma_1} \partial R_{\mu_2 \rho_2 \nu_2 \sigma_2}} \right)_t K_{l_1 \rho_1 \sigma_1} K_{l_2 \rho_2 \sigma_2} \\ &\quad \times \left[(n_{\mu_1 \mu_2} n_{\nu_1 \nu_2} - \epsilon_{\mu_1 \mu_2} \epsilon_{\nu_1 \nu_2}) n^{l_1 l_2} + (n_{\mu_1 \mu_2} \epsilon_{\nu_1 \nu_2} + \epsilon_{\mu_1 \mu_2} n_{\nu_1 \nu_2}) \epsilon^{l_1 l_2} \right] (\star 1) \\ &= 2\pi \times 4 \left\{ -2\mathbf{R}_N^2 \wedge \boldsymbol{\sigma} - 2\mathbf{R}_N \wedge \text{tr}[\mathbf{r}\mathbf{w}] - 2\text{tr}[\mathbf{r}^2\boldsymbol{\sigma}] - 4t\mathbf{R}_N \wedge \boldsymbol{\sigma}^2 - 8t\boldsymbol{\sigma} \wedge \text{tr}[\mathbf{r}\mathbf{w}] - 6t^2\boldsymbol{\sigma}^3 \right. \\ &\quad \left. + 4t(-i\epsilon_{ab})(\hat{D}\mathbf{K}^b) \cdot \mathbf{w} \cdot (\hat{D}\mathbf{K}^a) + 4t\boldsymbol{\sigma} \wedge d[\mathbf{K}^a \cdot (\hat{D}\mathbf{K}_a)] - 3td[\boldsymbol{\sigma}(\mathbf{K}^c \cdot \hat{D}\mathbf{K}_c)] \right\}, \end{aligned} \quad (\text{C.20})$$

after integrating over $\int_0^1 dt$, we obtain S_{ent}^{KK} as

$$\begin{aligned} S_{\text{ent}}^{KK} &= 2\pi \times 4 \int_{\Sigma \times I} \left\{ -2\boldsymbol{\sigma}^3 - 3\mathbf{R}_N \wedge \boldsymbol{\sigma}^2 + \frac{1}{2}(\hat{D}\mathbf{K}^a) \cdot (\hat{D}\mathbf{K}_a) \wedge \boldsymbol{\sigma} - 3\boldsymbol{\sigma} \wedge \text{tr}[\mathbf{r}\mathbf{w}] - 2\mathbf{R}_N^2 \wedge \boldsymbol{\sigma} \right. \\ &\quad \left. + \frac{3}{2}(\hat{D}\mathbf{K}^a) \cdot \boldsymbol{\sigma} \cdot (\hat{D}\mathbf{K}_a) + (-i\epsilon_{ab})\frac{1}{2}(\hat{D}\mathbf{K}^b) \cdot \mathbf{w} \cdot (\hat{D}\mathbf{K}^a) - 2\text{tr}[\mathbf{r}^2\boldsymbol{\sigma}] - 2\mathbf{R}_N \wedge \text{tr}[\mathbf{r}\mathbf{w}] \right\}. \end{aligned} \quad (\text{C.21})$$

Combining these two contribution, we then obtain

$$\begin{aligned} S_{\text{ent,CS}}^{7d, \text{single}} &= 16\pi \int_{\Sigma \times I} \left\{ \mathbf{R}_N^3 - \mathbf{R}_N \wedge d[\mathbf{K}^a \cdot (\hat{D}\mathbf{K}_a)] + \frac{3}{2}\mathbf{R}_N \wedge \boldsymbol{\sigma}^2 - \frac{3}{4}\boldsymbol{\sigma} \wedge (\hat{D}\mathbf{K}^a) \cdot (\hat{D}\mathbf{K}_a) \right. \\ &\quad - \frac{3}{2}\boldsymbol{\sigma} \wedge \text{tr}[\mathbf{r}\mathbf{w}] + \frac{1}{2}(-i\epsilon_{ab})(\hat{D}\mathbf{K}^b) \cdot \mathbf{r} \cdot (\hat{D}\mathbf{K}^a) + \mathbf{R}_N \wedge \text{tr}[\mathbf{r}\mathbf{w}] - \text{tr}[\mathbf{r}^2\boldsymbol{\sigma}] \\ &\quad \left. - \frac{3}{4}(-i\epsilon_{ab})(\hat{D}\mathbf{K}^b) \cdot \boldsymbol{\omega} \cdot (\hat{D}\mathbf{K}^a) + \frac{3}{4}(\hat{D}\mathbf{K}^a) \cdot \boldsymbol{\sigma} \cdot (\hat{D}\mathbf{K}_a) \right\}. \end{aligned} \quad (\text{C.22})$$

The final step is to rewrite this integrand in the form of $d(\dots)$. To do this, we first notice that

$$\begin{aligned} \mathbf{R}_N^3 &= d[\Gamma_N \wedge \mathbf{R}_N^2], \\ \mathbf{R}_N(\hat{D}\mathbf{K}^a) \cdot (\hat{D}\mathbf{K}_a) - 2\mathbf{R}_N^2 \wedge \boldsymbol{\sigma} &= d[\mathbf{R}_N \wedge \mathbf{K}^a \cdot (\hat{D}\mathbf{K}_a)] - 2\mathbf{R}_N \wedge \text{tr}[\mathbf{r}\mathbf{w}]. \end{aligned} \quad (\text{C.23})$$

Here we have used²⁴

$$(\hat{D}\mathbf{K}^a) \cdot (\hat{D}\mathbf{K}_a) = d[\mathbf{K}^a \cdot (\hat{D}\mathbf{K}_a)] - 2\text{tr}[\mathbf{r}\mathbf{w}] + 2\mathbf{R}_N \wedge \boldsymbol{\sigma}, \quad (\text{C.25})$$

which follows from

$$D^2\mathbf{K}_{ai} = \mathbf{r}_i^j \wedge \mathbf{K}_{aj}, \quad \hat{D}^2\mathbf{K}_{ai} = \mathbf{r}_i^j \wedge \mathbf{K}_{aj} + i\epsilon_{ab}\mathbf{R}_N \wedge \mathbf{K}^b_i. \quad (\text{C.26})$$

Furthermore, we can also have

$$\begin{aligned} d\boldsymbol{\sigma} &= i\epsilon^{ab}(\mathbf{K}_b \cdot D\mathbf{K}_a), \\ D\boldsymbol{\sigma}_{ij} &= \frac{1}{2}i\epsilon^{ab}[(\hat{D}\mathbf{K}_{ai}) \wedge \mathbf{K}_{bj} + (\hat{D}\mathbf{K}_{aj}) \wedge \mathbf{K}_{bi}], \\ D\mathbf{w}_{ij} &= \frac{1}{2}\hat{D}\mathbf{K}^a_i \wedge \mathbf{K}_{aj} - \frac{1}{2}\mathbf{K}^a_i \wedge \hat{D}\mathbf{K}_{aj}, \end{aligned} \quad (\text{C.27})$$

and²⁵

$$\begin{aligned} (d\boldsymbol{\sigma}) \wedge (\mathbf{K}^c \cdot \hat{D}\mathbf{K}_c) &= i\epsilon^{ab}(\mathbf{K}_b \cdot \hat{D}\mathbf{K}_a)(\mathbf{K}^c \cdot \hat{D}\mathbf{K}_c) \\ &= -(\hat{D}\mathbf{K}_c \cdot \boldsymbol{\sigma} \cdot \hat{D}\mathbf{K}^c) - i\epsilon^{ab}(\hat{D}\mathbf{K}_b \cdot \mathbf{w} \cdot \hat{D}\mathbf{K}_a). \end{aligned} \quad (\text{C.28})$$

Using all these useful identities, we can rewrite Eq. (C.22) to obtain the final expression for the holographic entanglement entropy formula:

$$\begin{aligned} S_{\text{ent,CS}}^{7d, \text{single}} &= 2\pi \times 4 \int_{\Sigma \times I} d \left[2\Gamma_N \wedge \mathbf{R}_N^2 - 2\mathbf{R}_N \wedge (\mathbf{K}^a \cdot \hat{D}\mathbf{K}_a) - \frac{3}{2}\boldsymbol{\sigma} \wedge (\mathbf{K}^a \cdot \hat{D}\mathbf{K}_a) + i\epsilon^{ab}(\mathbf{K}_a \cdot \mathbf{r} \cdot \hat{D}\mathbf{K}_b) \right] \\ &= 2(2\pi) \times 4 \int_{\Sigma} \left[\Gamma_N \wedge \mathbf{R}_N^2 - \left(\mathbf{R}_N + \frac{3}{4}\boldsymbol{\sigma} \right) \wedge (\mathbf{K}^a \cdot \hat{D}\mathbf{K}_a) + \frac{1}{2}i\epsilon^{ab}(\mathbf{K}_a \cdot \mathbf{r} \cdot \hat{D}\mathbf{K}_b) \right]. \end{aligned} \quad (\text{C.29})$$

It is straightforward to see that this indeed is equivalent to Eq. (1.5) by using the conversion formulas summarized in Appendix C.6.

²⁴More generally

$$\begin{aligned} (\hat{D}\mathbf{K}^b) \cdot (\hat{D}\mathbf{K}_a) &= \hat{D}[\mathbf{K}^b \cdot (\hat{D}\mathbf{K}_a)] + (\mathbf{K}^b \cdot \mathbf{r} \cdot \mathbf{K}_a) + \delta_a^b \mathbf{R}_N \wedge \boldsymbol{\sigma}, \\ (\hat{D}\mathbf{K}^a_i) \wedge (\hat{D}\mathbf{K}_{aj}) &= D[\mathbf{K}^a_i \wedge (\hat{D}\mathbf{K}_{aj})] - 2(\mathbf{r}\mathbf{w})_{ji} + 2\mathbf{R}_N \wedge \boldsymbol{\sigma}_{ji}, \\ i\epsilon^a_b(\hat{D}\mathbf{K}^b_i) \wedge (\hat{D}\mathbf{K}_{aj}) &= D[i\epsilon^a_b(\mathbf{K}^b_i) \wedge (\hat{D}\mathbf{K}_{aj})] - 2(\mathbf{r}\boldsymbol{\sigma})_{ji} + 2\mathbf{R}_N \wedge \mathbf{w}_{ji}. \end{aligned} \quad (\text{C.24})$$

²⁵The easiest way to show this is to take the complex $x^a = (z, \bar{z})$ coordinates (see Appendix C.6).

C.5 $\mathcal{P}_{\text{anom}} = \text{tr}(\mathbf{R}^2) \wedge \text{tr}(\mathbf{R}^2)$

As the final example, here we deal with the anomaly polynomial corresponding to 7d double-trace gravitational Chern-Simons term.

First of all, the Wald term is evaluated to have

$$S_{\text{ent}}^{\text{Wald}} = 2\pi \times 8 \int_{\Sigma \times I} \left[\mathbf{R}_N \wedge \text{tr}(\mathbf{R}^2) - \boldsymbol{\sigma} \wedge (2\mathbf{R}_N^2 - \text{tr}[\mathbf{r}^2]) - 2\boldsymbol{\sigma} \wedge d[\mathbf{K}^a \cdot (\hat{D}\mathbf{K}_a)] \right]. \quad (\text{C.30})$$

Secondly, for the S_{ent}^{KK} term, we first compute the integrand as

$$\begin{aligned} & 4\pi \left(\frac{\partial^2 L}{\partial R_{\mu_1 \rho_1 \nu_1 \sigma_1} \partial R_{\mu_2 \rho_2 \nu_2 \sigma_2}} \right)_t K_{l_1 \rho_1 \sigma_1} K_{l_2 \rho_2 \sigma_2} \\ & \times \left[(n_{\mu_1 \mu_2} n_{\nu_1 \nu_2} - \epsilon_{\mu_1 \mu_2} \epsilon_{\nu_1 \nu_2}) n^{l_1 l_2} + (n_{\mu_1 \mu_2} \epsilon_{\nu_1 \nu_2} + \epsilon_{\mu_1 \mu_2} n_{\nu_1 \nu_2}) \epsilon^{l_1 l_2} \right] (\star 1) \\ & = 2\pi \times 8 \times \left\{ \boldsymbol{\sigma} \wedge (-2\mathbf{R}_N^2 - \text{tr}[\mathbf{r}^2]) + 4t\boldsymbol{\sigma} \wedge d[\mathbf{K}^a \cdot (\hat{D}\mathbf{K}_a)] - 2td[\boldsymbol{\sigma} \wedge (\mathbf{K}^c \cdot \hat{D}\mathbf{K}_c)] \right\}. \end{aligned} \quad (\text{C.31})$$

Thus, by integrating this, we obtain

$$S_{\text{ent}}^{KK} = 2\pi \times 8 \int_{\Sigma \times I} \left[\boldsymbol{\sigma} \wedge (-2\mathbf{R}_N^2 - \text{tr}[\mathbf{r}^2]) + 2\boldsymbol{\sigma} \wedge d[\mathbf{K}^a \cdot (\hat{D}\mathbf{K}_a)] - d[\boldsymbol{\sigma} \wedge (\mathbf{K}^c \cdot \hat{D}\mathbf{K}_c)] \right]. \quad (\text{C.32})$$

Combining these two contributions and after some massaging the expression, we finally get the holographic entanglement entropy formula as

$$\begin{aligned} S_{\text{ent,CS}}^{7d, \text{double}} &= 16\pi \times 2 \int_{\Sigma \times I} d \left[\boldsymbol{\Gamma}_N \wedge \mathbf{R}_N^2 + \frac{1}{2} \boldsymbol{\Gamma}_N \wedge \text{tr}[\mathbf{r}^2] - \left(\mathbf{R}_N + \frac{1}{2} \boldsymbol{\sigma} \right) \wedge (\mathbf{K}^a \cdot \hat{D}\mathbf{K}_a) \right] \\ &= 16\pi \times 2 \int_{\Sigma} \left[\boldsymbol{\Gamma}_N \wedge \mathbf{R}_N^2 + \frac{1}{2} \boldsymbol{\Gamma}_N \wedge \text{tr}[\mathbf{r}^2] - \left(\mathbf{R}_N + \frac{1}{2} \boldsymbol{\sigma} \right) \wedge (\mathbf{K}^a \cdot \hat{D}\mathbf{K}_a) \right], \end{aligned} \quad (\text{C.33})$$

which reproduces Eq. (1.6) after using the conversion formulas summarized in Appendix C.6.

C.6 Formulas for Conversion

In this part, we summarize some dictionaries for converting the notation in this Appendix to that used in the rest part of this paper.

We first note that, when we take (z, \bar{z}) -coordinate for the directions orthogonal to $\tilde{\Sigma} = \Sigma \times I$, we can explicitly write down the induced metric on this two-dimensional plane and the binormal at $\tilde{\Sigma}$ as

$$n_{z\bar{z}} = 1/2, \quad n^{z\bar{z}} = 2, \quad \epsilon_{z\bar{z}} = \frac{i}{2}, \quad \epsilon^{\bar{z}z} = 2i, \quad \epsilon_z^z = -\epsilon_{\bar{z}}^{\bar{z}} = -i. \quad (\text{C.34})$$

The extrinsic curvatures are:

$$\mathbf{K} \equiv \mathbf{K}_z, \quad \bar{\mathbf{K}} \equiv \mathbf{K}_{\bar{z}}. \quad (\text{C.35})$$

Then, by using these, we can also have

$$\begin{aligned}
\hat{D}\mathbf{K}_j &= D\mathbf{K}_j - \mathbf{\Gamma}_N \wedge \mathbf{K}_j, & \hat{D}\bar{\mathbf{K}}_j &= D\bar{\mathbf{K}}_j + \mathbf{\Gamma}_N \wedge \bar{\mathbf{K}}_j, \\
\sigma_{ij} &= \frac{1}{2}i\epsilon^{ab}\mathbf{K}_{ai} \wedge \mathbf{K}_{bj} = \mathbf{K}_i \wedge \bar{\mathbf{K}}_j - \bar{\mathbf{K}}_i \wedge \mathbf{K}_j, & \boldsymbol{\sigma} &= \boldsymbol{\sigma}^i{}_i = -2(\bar{\mathbf{K}} \cdot \mathbf{K}), \\
w_{ij} &= \frac{1}{2}\mathbf{K}^a{}_i \wedge \mathbf{K}_{aj} = \bar{\mathbf{K}}_i \wedge \mathbf{K}_j + \mathbf{K}_i \wedge \bar{\mathbf{K}}_j.
\end{aligned} \tag{C.36}$$

In the end, we also summarize some extra conversion formulas that are useful for our computation:

$$(\mathbf{K}^a \cdot \hat{D}\mathbf{K}_a) = 2(\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}} + \bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}), \tag{C.37}$$

$$\frac{1}{2}i\epsilon^{ab}(\mathbf{K}_a \cdot \mathbf{r} \cdot \hat{D}\mathbf{K}_b) = \hat{D}\mathbf{K} \cdot \mathbf{r} \cdot \bar{\mathbf{K}} - \hat{D}\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{K}. \tag{C.38}$$

D 7d Single-Trace Gravitational Chern-Simons Term

In this section, we will directly evaluate the 7d single-trace gravitational Chern-Simons term on the regularized cone background. We will reproduce the result obtained from the anomaly-polynomial-based computation in §3.

Before the evaluation, let us recall that the 7d single-trace gravitational Chern-Simons term is given by

$$\begin{aligned}
\mathbf{I}_{\text{CS}}^{7d, \text{single}} &= 4 \int_0^1 dt \operatorname{tr} [\mathbf{\Gamma} \wedge (t\mathbf{R} + t(t-1)\mathbf{\Gamma}^2)^3] \\
&= 4 \int_0^1 dt \operatorname{tr} \left[t^3(t-1)^3\mathbf{\Gamma}^7 + 3t^3(t-1)^2\mathbf{\Gamma}^5 \wedge \mathbf{R} \right. \\
&\quad \left. + t^3(t-1)(2\mathbf{\Gamma}^3 \wedge \mathbf{R}^2 + \mathbf{\Gamma} \wedge \mathbf{R} \wedge \mathbf{\Gamma}^2 \wedge \mathbf{R}) + t^3\mathbf{\Gamma} \wedge \mathbf{R}^3 \right] \\
&= \operatorname{tr} \left[-\frac{1}{35}\mathbf{\Gamma}^7 + \frac{1}{5}\mathbf{\Gamma}^5 \wedge \mathbf{R} - \frac{1}{5}\mathbf{\Gamma} \wedge \mathbf{R} \wedge \mathbf{\Gamma}^2 \wedge \mathbf{R} - \frac{2}{5}\mathbf{\Gamma}^3 \wedge \mathbf{R}^2 + \mathbf{\Gamma} \wedge \mathbf{R}^3 \right]. \tag{D.1}
\end{aligned}$$

Thus, what we need to evaluate is $\operatorname{tr}(\mathbf{\Gamma}^7)$, $\operatorname{tr}(\mathbf{\Gamma}^5 \wedge \mathbf{R})$, $\operatorname{tr}(\mathbf{\Gamma} \wedge \mathbf{R} \wedge \mathbf{\Gamma}^2 \wedge \mathbf{R})$, $\operatorname{tr}(\mathbf{\Gamma}^3 \wedge \mathbf{R}^2)$ and $\operatorname{tr}(\mathbf{\Gamma} \wedge \mathbf{R}^3)$ on the regularized cone background. We will deal with each term separately and eventually combine them to obtain the holographic entanglement entropy formula.

D.1 Term-by-term Contributions

We now compute the contributions of each term in $\mathbf{I}_{\text{CS}}^{7d, \text{single}}$ one by one.

D.1.1 $\operatorname{tr}(\mathbf{\Gamma}^7)$

(1) $\partial\bar{\partial}A$ Term

Since the connection one-form does not contain terms of the form $\partial\bar{\partial}A$, we conclude that there is no this type of contribution:

$$\int_{\text{Cone}_{\Sigma, n}} \operatorname{tr}(\mathbf{\Gamma}^7)|_{\partial\bar{\partial}A} = 0. \tag{D.2}$$

(2) $\partial A \bar{\partial} A$ Term

In the connection one-form, $\partial A dz$ (resp. $\bar{\partial} A d\bar{z}$) shows up only in Γ^z_z (resp. only in $\Gamma^{\bar{z}}_{\bar{z}}$). Therefore, the potential nontrivial contributions of this form are

$$\begin{aligned} & {}^{(d)}\Gamma^z_z{}^{(0)}(\Gamma^2)^z_{\bar{z}}{}^{(d)}\Gamma^{\bar{z}}_{\bar{z}}{}^{(0)}(\Gamma^3)^{\bar{z}}_z, & {}^{(d)}\Gamma^z_z{}^{(0)}(\Gamma^3)^z_{\bar{z}}{}^{(d)}\Gamma^{\bar{z}}_{\bar{z}}{}^{(0)}(\Gamma^2)^{\bar{z}}_z, \\ & {}^{(d)}\Gamma^z_z{}^{(0)}\Gamma^z_{\bar{z}}{}^{(d)}\Gamma^{\bar{z}}_{\bar{z}}{}^{(0)}(\Gamma^4)^{\bar{z}}_z, & {}^{(d)}\Gamma^z_z{}^{(0)}(\Gamma^4)^z_{\bar{z}}{}^{(d)}\Gamma^{\bar{z}}_{\bar{z}}{}^{(0)}\Gamma^{\bar{z}}_z, \end{aligned} \quad (\text{D.3})$$

but all of them are zero since ${}^{(0)}\Gamma^z_{\bar{z}} = {}^{(0)}\Gamma^{\bar{z}}_z = {}^{(0)}(\Gamma^2)^z_{\bar{z}} = {}^{(0)}(\Gamma^2)^{\bar{z}}_z = 0$. We thus conclude that there is no contribution of this type:

$$\int_{\text{Cone}_{\Sigma,n}} \text{tr}(\Gamma^7)|_{\partial A \bar{\partial} A} = 0. \quad (\text{D.4})$$

(3) ∂A or $\bar{\partial} A$ Term

It is trivial that there is no this type of contribution since $\text{tr}(\Gamma^7)$ does not contain $d\Gamma$:

$$\int_{\text{Cone}_{\Sigma,n}} \text{tr}(\Gamma^7)|_{\partial A, \bar{\partial} A} = 0. \quad (\text{D.5})$$

Summary for $\text{tr}(\Gamma^7)$

From the above, by combining Eq. (D.2), (D.4) and (D.5), we conclude that $\text{tr}(\Gamma^7)$ does not generate any order- ϵ contribution:

$$\left[\int_{\text{Cone}_{\Sigma,n}} \text{tr}(\Gamma^7) \right]_{\epsilon} = 0. \quad (\text{D.6})$$

D.1.2 $\text{tr}(\Gamma^5 \wedge \mathbf{R})$

As a next step, we evaluate $\text{tr}(\Gamma^5 \wedge \mathbf{R}) = (\Gamma^2 \mathbf{R})^\mu{}_\nu \wedge (\Gamma^3)^\nu{}_\mu$ on the regularized cone background.

(1) $\partial \bar{\partial} A$ Term

From the expression of $\Gamma^2 \mathbf{R}$ and Γ^3 summarized in Appendix A, we first note that there are the following potential nontrivial contributions:

$$\begin{aligned} & {}^{(d^2)}(\Gamma^2 \mathbf{R})^z_z{}^{(0)}(\Gamma^3)^z_z, & {}^{(d^2)}(\Gamma^2 \mathbf{R})^{\bar{z}}_{\bar{z}}{}^{(0)}(\Gamma^3)^{\bar{z}}_{\bar{z}}, \\ & {}^{(d^2)}(\Gamma^2 \mathbf{R})^i_z{}^{(0)}(\Gamma^3)^z_i, & {}^{(d^2)}(\Gamma^2 \mathbf{R})^i_{\bar{z}}{}^{(0)}(\Gamma^3)^{\bar{z}}_i. \end{aligned} \quad (\text{D.7})$$

Each term is directly computed as follows:

$$\begin{aligned} & {}^{(d^2)}(\Gamma^2 \mathbf{R})^z_z{}^{(0)}(\Gamma^3)^z_z = (-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge (-2e^{-2A})^2 [2\Gamma_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 + (\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K})], \\ & {}^{(d^2)}(\Gamma^2 \mathbf{R})^{\bar{z}}_{\bar{z}}{}^{(0)}(\Gamma^3)^{\bar{z}}_{\bar{z}} = (-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge (-2e^{-2A})^2 [2\Gamma_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 + (\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}})], \\ & {}^{(d^2)}(\Gamma^2 \mathbf{R})^i_z{}^{(0)}(\Gamma^3)^z_i = (-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge \left[(-2e^{-2A})[2\Gamma_N \wedge (\bar{\mathbf{K}} \cdot \gamma^2 \cdot \mathbf{K}) + (\bar{\mathbf{K}} \cdot \gamma^3 \cdot \mathbf{K})] \right. \\ & \quad \left. + (-2e^{-2A})^2[\Gamma_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 + (\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K})] \right], \\ & {}^{(d^2)}(\Gamma^2 \mathbf{R})^i_{\bar{z}}{}^{(0)}(\Gamma^3)^{\bar{z}}_i = (-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge \left[(-2e^{-2A})[2\Gamma_N \wedge (\mathbf{K} \cdot \gamma^2 \cdot \bar{\mathbf{K}}) - (\mathbf{K} \cdot \gamma^3 \cdot \bar{\mathbf{K}})] \right. \\ & \quad \left. + (-2e^{-2A})^2[\Gamma_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 + (\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}})] \right]. \end{aligned} \quad (\text{D.8})$$

By summing them and doing the integration, we finally obtain the total of this type of contribution:

$$\begin{aligned}
& \int_{\text{Cone}_{\Sigma, n}} \text{tr}(\mathbf{\Gamma}^5 \wedge \mathbf{R})|_{\partial \bar{\partial} A} \\
&= (4\pi\epsilon) \int_{\Sigma} \left[12\mathbf{\Gamma}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 - 2\mathbf{\Gamma}_N \wedge [(\bar{\mathbf{K}} \cdot \gamma^2 \cdot \mathbf{K}) + (\mathbf{K} \cdot \gamma^2 \cdot \bar{\mathbf{K}})] \right. \\
&\quad \left. + 4(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge [(\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K}) + (\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}})] - [(\bar{\mathbf{K}} \cdot \gamma^3 \cdot \mathbf{K}) - (\mathbf{K} \cdot \gamma^3 \cdot \bar{\mathbf{K}})] \right].
\end{aligned} \tag{D.9}$$

(2) $\partial A \bar{\partial} A$ Term

For this type of contribution, from the expression of $\mathbf{\Gamma}^2 \mathbf{R}$ and $\mathbf{\Gamma}^3$ summarized in Appendix A, potential nontrivial possibilities are as follows:

$$\begin{aligned}
& {}^{(dd)}(\mathbf{\Gamma}^2 \mathbf{R})^i_j {}^{(0)}(\mathbf{\Gamma}^3)^j_i, \\
& {}^{(d)}(\mathbf{\Gamma}^2 \mathbf{R})^z_z {}^{(0)}(\mathbf{\Gamma}^3)^z_z, \quad {}^{(d)}(\mathbf{\Gamma}^2 \mathbf{R})^{\bar{z}}_{\bar{z}} {}^{(d)}(\mathbf{\Gamma}^3)^{\bar{z}}_{\bar{z}}, \\
& {}^{(d)}(\mathbf{\Gamma}^2 \mathbf{R})^z_j {}^{(d)}(\mathbf{\Gamma}^3)^j_z, \quad {}^{(d)}(\mathbf{\Gamma}^2 \mathbf{R})^{\bar{z}}_j {}^{(d)}(\mathbf{\Gamma}^3)^j_{\bar{z}}, \\
& {}^{(d)}(\mathbf{\Gamma}^2 \mathbf{R})^i_z {}^{(d)}(\mathbf{\Gamma}^3)^z_i, \quad {}^{(d)}(\mathbf{\Gamma}^2 \mathbf{R})^i_{\bar{z}} {}^{(d)}(\mathbf{\Gamma}^3)^{\bar{z}}_i, \\
& {}^{(d)}(\mathbf{\Gamma}^2 \mathbf{R})^i_j {}^{(d)}(\mathbf{\Gamma}^3)^j_i.
\end{aligned} \tag{D.10}$$

We can evaluate each term as

$$\begin{aligned}
& {}^{(dd)}(\mathbf{\Gamma}^2 \mathbf{R})^i_j {}^{(0)}(\mathbf{\Gamma}^3)^j_i \\
&= (-4\partial A \bar{\partial} A dz \wedge d\bar{z}) \wedge \left[(-2e^{-2A})[(\bar{\mathbf{K}} \cdot \gamma^3 \cdot \mathbf{K}) - (\mathbf{K} \cdot \gamma^3 \cdot \bar{\mathbf{K}})] \right. \\
&\quad \left. + (-2e^{-2A})^2 \left[2\mathbf{\Gamma}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 + 2(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge [(\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K}) + (\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}})] \right] \right], \\
& {}^{(d)}(\mathbf{\Gamma}^2 \mathbf{R})^z_z {}^{(d)}(\mathbf{\Gamma}^3)^z_z = 0, \quad {}^{(d)}(\mathbf{\Gamma}^2 \mathbf{R})^{\bar{z}}_{\bar{z}} {}^{(d)}(\mathbf{\Gamma}^3)^{\bar{z}}_{\bar{z}} = 0, \\
& {}^{(d)}(\mathbf{\Gamma}^2 \mathbf{R})^z_j {}^{(d)}(\mathbf{\Gamma}^3)^j_z = (-4\partial \bar{\partial} A dz \wedge d\bar{z}) \wedge (-2e^{-2A})^2 (\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K}), \\
& {}^{(d)}(\mathbf{\Gamma}^2 \mathbf{R})^{\bar{z}}_j {}^{(d)}(\mathbf{\Gamma}^3)^j_{\bar{z}} = (-4\partial \bar{\partial} A dz \wedge d\bar{z}) \wedge (-2e^{-2A})^2 (\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}}), \\
& {}^{(d)}(\mathbf{\Gamma}^2 \mathbf{R})^i_z {}^{(d)}(\mathbf{\Gamma}^3)^z_i = 0, \quad {}^{(d)}(\mathbf{\Gamma}^2 \mathbf{R})^i_{\bar{z}} {}^{(d)}(\mathbf{\Gamma}^3)^{\bar{z}}_i = 0, \\
& {}^{(d)}(\mathbf{\Gamma}^2 \mathbf{R})^i_j {}^{(d)}(\mathbf{\Gamma}^3)^j_i \\
&= (-4\partial A \bar{\partial} A dz \wedge d\bar{z}) \wedge (-2e^{-2A})^2 \left[-2\mathbf{\Gamma}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 - (\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge [(\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K}) + (\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}})] \right]. \tag{D.11}
\end{aligned}$$

By summing all of them and then integrating, we conclude that the total of this type of

contribution is given by

$$\begin{aligned} & \int_{\text{Cone}_{\Sigma,n}} \text{tr}(\mathbf{\Gamma}^5 \wedge \mathbf{R})|_{\partial A \bar{\partial} A} \\ &= (4\pi\epsilon) \int_{\Sigma} \left[-[(\bar{\mathbf{K}} \cdot \gamma^3 \cdot \mathbf{K}) - (\mathbf{K} \cdot \gamma^3 \cdot \bar{\mathbf{K}})] + 2(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge [(\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K}) + (\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}})] \right]. \end{aligned} \quad (\text{D.12})$$

(3) ∂A or $\bar{\partial} A$ Term

By using Appendix A, we can see that all of this type of terms contain \mathbf{K} and $\bar{\mathbf{K}}$. Thus, from the argument of §4.3, it follows that there is no contribution of this type:

$$\int_{\text{Cone}_{\Sigma,n}} \text{tr}(\mathbf{\Gamma}^5 \wedge \mathbf{R})|_{\partial A, \bar{\partial} A} = 0. \quad (\text{D.13})$$

Summary for $\text{tr}(\mathbf{\Gamma}^5 \wedge \mathbf{R})$

By summing up Eqs. (D.9), (D.12) and (D.13), we finally obtain the order- ϵ contribution from $\text{tr}(\mathbf{\Gamma}^5 \wedge \mathbf{R})$ as

$$\begin{aligned} & \left[\int_{\text{Cone}_{\Sigma,n}} \text{tr}(\mathbf{\Gamma}^5 \wedge \mathbf{R}) \right]_{\epsilon} \\ &= (4\pi\epsilon) \int_{\Sigma} \left[12\mathbf{\Gamma}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 - 2\mathbf{\Gamma}_N \wedge [(\bar{\mathbf{K}} \cdot \gamma^2 \cdot \mathbf{K}) + (\mathbf{K} \cdot \gamma^2 \cdot \bar{\mathbf{K}})] \right. \\ & \quad \left. + 6(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge [(\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K}) + (\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}})] - 2[(\bar{\mathbf{K}} \cdot \gamma^3 \cdot \mathbf{K}) - (\mathbf{K} \cdot \gamma^3 \cdot \bar{\mathbf{K}})] \right]. \end{aligned} \quad (\text{D.14})$$

D.1.3 $\text{tr}(\mathbf{\Gamma} \wedge \mathbf{R} \wedge \mathbf{\Gamma}^2 \wedge \mathbf{R})$

Here we will evaluate $\text{tr}(\mathbf{\Gamma} \wedge \mathbf{R} \wedge \mathbf{\Gamma}^2 \wedge \mathbf{R}) = (\mathbf{\Gamma}^2 \mathbf{R})^{\mu}_{\nu} \wedge (\mathbf{\Gamma} \mathbf{R})^{\nu}_{\mu}$ on the regularized cone background.

(1) $\partial \bar{\partial} A$ Term

The followings are the potential nontrivial terms of this type:

$$\begin{aligned} & (d^2)(\mathbf{\Gamma}^2 \mathbf{R})^z_z (0)(\mathbf{\Gamma} \mathbf{R})^z_z, & (d^2)(\mathbf{\Gamma}^2 \mathbf{R})^{\bar{z}}_{\bar{z}} (0)(\mathbf{\Gamma} \mathbf{R})^{\bar{z}}_{\bar{z}}, \\ & (d^2)(\mathbf{\Gamma}^2 \mathbf{R})^i_z (0)(\mathbf{\Gamma} \mathbf{R})^z_i, & (d^2)(\mathbf{\Gamma}^2 \mathbf{R})^i_{\bar{z}} (0)(\mathbf{\Gamma} \mathbf{R})^{\bar{z}}_i, \\ & (0)(\mathbf{\Gamma}^2 \mathbf{R})^z_z (d^2)(\mathbf{\Gamma} \mathbf{R})^z_z, & (0)(\mathbf{\Gamma}^2 \mathbf{R})^{\bar{z}}_{\bar{z}} (d^2)(\mathbf{\Gamma} \mathbf{R})^{\bar{z}}_{\bar{z}}, \\ & (0)(\mathbf{\Gamma}^2 \mathbf{R})^z_i (d^2)(\mathbf{\Gamma} \mathbf{R})^i_z, & (0)(\mathbf{\Gamma}^2 \mathbf{R})^{\bar{z}}_i (d^2)(\mathbf{\Gamma} \mathbf{R})^i_{\bar{z}}, \end{aligned} \quad (\text{D.15})$$

where each term is computed as

$$\begin{aligned}
& (d^2)(\Gamma^2 \mathbf{R})^z_z{}^{(0)}(\Gamma \mathbf{R})^z_z \\
&= (-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge \left[(-2e^{-2A})\Gamma_N \wedge \mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) + (-2e^{-2A})^2[\Gamma_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 + (\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K})] \right], \\
& (d^2)(\Gamma^2 \mathbf{R})^{\bar{z}}_{\bar{z}}{}^{(0)}(\Gamma \mathbf{R})^{\bar{z}}_{\bar{z}} \\
&= (-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge \left[(-2e^{-2A})\Gamma_N \wedge \mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) + (-2e^{-2A})^2[\Gamma_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 + (\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}})] \right], \\
& (d^2)(\Gamma^2 \mathbf{R})^i_z{}^{(0)}(\Gamma \mathbf{R})^z_i \\
&= (-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge \left[(-2e^{2A}) \left[\Gamma_N \wedge [(\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{K}) + (\hat{D}\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K})] + (\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \gamma \cdot \mathbf{K}) \right] \right. \\
&\quad \left. + (-2e^{2A})^2[\Gamma_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 + (\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K})] \right], \\
& (d^2)(\Gamma^2 \mathbf{R})^i_{\bar{z}}{}^{(0)}(\Gamma \mathbf{R})^{\bar{z}}_i \\
&= (-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge \left[(-2e^{2A}) \left[\Gamma_N \wedge [(\mathbf{K} \cdot \mathbf{r} \cdot \bar{\mathbf{K}}) + (\hat{D}\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}})] - (\mathbf{K} \cdot \mathbf{r} \cdot \gamma \cdot \bar{\mathbf{K}}) \right] \right. \\
&\quad \left. + (-2e^{2A})^2[\Gamma_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 + (\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}})] \right], \\
& (0)(\Gamma^2 \mathbf{R})^z_z{}^{(d^2)}(\Gamma \mathbf{R})^z_z \\
&= (-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge \left[(-2e^{-2A})[\Gamma_N \wedge \mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) + \Gamma_N \wedge (\bar{\mathbf{K}} \cdot \gamma \cdot \hat{D}\mathbf{K})] + (-2e^{-2A})^2\Gamma_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 \right], \\
& (0)(\Gamma^2 \mathbf{R})^{\bar{z}}_{\bar{z}}{}^{(d^2)}(\Gamma \mathbf{R})^{\bar{z}}_{\bar{z}} \\
&= (-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge \left[(-2e^{-2A})[\Gamma_N \wedge \mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) + \Gamma_N \wedge (\mathbf{K} \cdot \gamma \cdot \hat{D}\bar{\mathbf{K}})] + (-2e^{-2A})^2\Gamma_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 \right], \\
& (0)(\Gamma^2 \mathbf{R})^z_i{}^{(d^2)}(\Gamma \mathbf{R})^i_z \\
&= (-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge \left[(-2e^{-2A})[\Gamma_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{K}) + (\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{r} \cdot \mathbf{K})] \right. \\
&\quad \left. + (-2e^{-2A})^2 \left[(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge [(\hat{D}\bar{\mathbf{K}} \cdot \mathbf{K}) + (\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K})] + \Gamma_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 \right] \right], \\
& (0)(\Gamma^2 \mathbf{R})^{\bar{z}}_i{}^{(d^2)}(\Gamma \mathbf{R})^i_{\bar{z}} \\
&= (-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge \left[(-2e^{-2A})[\Gamma_N \wedge (\mathbf{K} \cdot \mathbf{r} \cdot \bar{\mathbf{K}}) - (\mathbf{K} \cdot \gamma \cdot \mathbf{r} \cdot \bar{\mathbf{K}})] \right. \\
&\quad \left. + (-2e^{-2A})^2 \left[(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge [(\hat{D}\mathbf{K} \cdot \bar{\mathbf{K}}) + (\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}})] + \Gamma_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 \right] \right]. \quad (\text{D.16})
\end{aligned}$$

Therefore, by summing all the contributions above and doing the integration, we finally obtain

$$\begin{aligned}
& \int_{\text{Cone}_{\Sigma,n}} \text{tr}(\mathbf{\Gamma} \wedge \mathbf{R} \wedge \mathbf{\Gamma}^2 \wedge \mathbf{R})|_{\partial\bar{\partial}A} \\
&= (4\pi\epsilon) \int_{\Sigma} \left[-4\mathbf{\Gamma}_N \wedge \mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) \right. \\
&\quad + \mathbf{\Gamma}_N \wedge \left[-2[(\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{K}) + (\mathbf{K} \cdot \mathbf{r} \cdot \bar{\mathbf{K}})] + 16(\bar{\mathbf{K}} \cdot \mathbf{K})^2 \right. \\
&\quad \left. - [(\hat{D}\mathbf{K} \cdot \gamma \cdot \mathbf{K}) + (\hat{D}\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}}) + (\bar{\mathbf{K}} \cdot \gamma \cdot \hat{D}\mathbf{K}) + (\mathbf{K} \cdot \gamma \cdot \hat{D}\bar{\mathbf{K}})] \right] \\
&\quad + 4(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge [(\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K}) + (\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}})] + 4(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge [(\hat{D}\bar{\mathbf{K}} \cdot \mathbf{K}) + (\hat{D}\mathbf{K} \cdot \bar{\mathbf{K}})] \\
&\quad \left. - [(\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \gamma \cdot \mathbf{K}) - (\mathbf{K} \cdot \mathbf{r} \cdot \gamma \cdot \bar{\mathbf{K}}) + (\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{r} \cdot \mathbf{K}) - (\mathbf{K} \cdot \gamma \cdot \mathbf{r} \cdot \bar{\mathbf{K}})] \right].
\end{aligned} \tag{D.17}$$

(2) $\partial A \bar{\partial} A$ Term

The nontrivial possibilities are as follows:

$$\begin{aligned}
& {}^{(dd)}(\mathbf{\Gamma}^2 \mathbf{R})^i{}_j {}^{(0)}(\mathbf{\Gamma} \mathbf{R})^j{}_i, \\
& {}^{(0)}(\mathbf{\Gamma}^2 \mathbf{R})^j{}_z {}^{(dd)}(\mathbf{\Gamma} \mathbf{R})^z{}_j, \quad {}^{(0)}(\mathbf{\Gamma}^2 \mathbf{R})^j{}_{\bar{z}} {}^{(dd)}(\mathbf{\Gamma} \mathbf{R})^{\bar{z}}{}_j, \\
& {}^{(d)}(\mathbf{\Gamma}^2 \mathbf{R})^z{}_z {}^{(d)}(\mathbf{\Gamma} \mathbf{R})^z{}_z, \quad {}^{(d)}(\mathbf{\Gamma}^2 \mathbf{R})^{\bar{z}}{}_{\bar{z}} {}^{(d)}(\mathbf{\Gamma} \mathbf{R})^{\bar{z}}{}_{\bar{z}}, \\
& {}^{(d)}(\mathbf{\Gamma}^2 \mathbf{R})^z{}_j {}^{(d)}(\mathbf{\Gamma} \mathbf{R})^j{}_z, \quad {}^{(d)}(\mathbf{\Gamma}^2 \mathbf{R})^{\bar{z}}{}_j {}^{(d)}(\mathbf{\Gamma} \mathbf{R})^j{}_{\bar{z}}, \\
& {}^{(d)}(\mathbf{\Gamma}^2 \mathbf{R})^i{}_z {}^{(d)}(\mathbf{\Gamma} \mathbf{R})^z{}_i, \quad {}^{(d)}(\mathbf{\Gamma}^2 \mathbf{R})^i{}_{\bar{z}} {}^{(d)}(\mathbf{\Gamma} \mathbf{R})^{\bar{z}}{}_i, \\
& {}^{(d)}(\mathbf{\Gamma}^2 \mathbf{R})^i{}_j {}^{(d)}(\mathbf{\Gamma} \mathbf{R})^j{}_i.
\end{aligned} \tag{D.18}$$

Each term above is computed as

$$\begin{aligned}
& {}^{(dd)}(\Gamma^2 \mathbf{R})^i_j {}^{(0)}(\Gamma \mathbf{R})^j_i \\
&= (-4\partial A \bar{\partial} A dz \wedge d\bar{z}) \wedge \left[(-2e^{-2A})[(\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{r} \cdot \mathbf{K}) - (\mathbf{K} \cdot \gamma \cdot \mathbf{r} \cdot \bar{\mathbf{K}})] \right. \\
&\quad \left. + (-2e^{-2A})^2(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge [(\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K}) + (\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}}) + (\hat{D}\bar{\mathbf{K}} \cdot \mathbf{K}) + (\hat{D}\mathbf{K} \cdot \bar{\mathbf{K}})] \right], \\
& {}^{(0)}(\Gamma^2 \mathbf{R})^j_z {}^{(dd)}(\Gamma \mathbf{R})^z_j \\
&= (-4\partial A \bar{\partial} A dz \wedge d\bar{z}) \wedge \left[(-2e^{-2A})[\Gamma_N \wedge \mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) + \mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K}) + (\bar{\mathbf{K}} \cdot \gamma^2 \cdot \hat{D}\mathbf{K})] \right. \\
&\quad \left. + (-2e^{-2A})^2 \left[\Gamma_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 + (\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge [(\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K}) + (\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K})] \right] \right], \\
& {}^{(0)}(\Gamma^2 \mathbf{R})^j_{\bar{z}} {}^{(dd)}(\Gamma \mathbf{R})^{\bar{z}}_j \\
&= (-4\partial A \bar{\partial} A dz \wedge d\bar{z}) \wedge \left[(-2e^{-2A})[\Gamma_N \wedge \mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) + \mathbf{R}_N \wedge (\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}}) - (\mathbf{K} \cdot \gamma^2 \cdot \hat{D}\bar{\mathbf{K}})] \right. \\
&\quad \left. + (-2e^{-2A})^2 \left[\Gamma_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 + (\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge [(\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}}) + (\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}})] \right] \right], \\
& {}^{(d)}(\Gamma^2 \mathbf{R})^z_z {}^{(d)}(\Gamma \mathbf{R})^z_z = 0, \quad {}^{(d)}(\Gamma^2 \mathbf{R})^{\bar{z}}_{\bar{z}} {}^{(d)}(\Gamma \mathbf{R})^{\bar{z}}_{\bar{z}} = 0, \\
& {}^{(d)}(\Gamma^2 \mathbf{R})^z_j {}^{(d)}(\Gamma \mathbf{R})^j_z = (-4\partial A \bar{\partial} A dz \wedge d\bar{z}) \wedge (-2e^{-2A})^2(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K}), \\
& {}^{(d)}(\Gamma^2 \mathbf{R})^{\bar{z}}_j {}^{(d)}(\Gamma \mathbf{R})^j_{\bar{z}} = (-4\partial A \bar{\partial} A dz \wedge d\bar{z}) \wedge (-2e^{-2A})^2(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}}), \\
& {}^{(d)}(\Gamma^2 \mathbf{R})^i_z {}^{(d)}(\Gamma \mathbf{R})^z_i \\
&= (-4\partial A \bar{\partial} A dz \wedge d\bar{z}) \wedge \left[(-2e^{-2A})[-\Gamma_N \wedge \mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) - \Gamma_N \wedge (\bar{\mathbf{K}} \cdot \gamma^2 \cdot \mathbf{K})] \right. \\
&\quad \left. + (-2e^{-2A})^2[-2\Gamma_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2] \right], \\
& {}^{(d)}(\Gamma^2 \mathbf{R})^i_{\bar{z}} {}^{(d)}(\Gamma \mathbf{R})^{\bar{z}}_i \\
&= (-4\partial A \bar{\partial} A dz \wedge d\bar{z}) \wedge \left[(-2e^{-2A})[-\Gamma_N \wedge \mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) - \Gamma_N \wedge (\mathbf{K} \cdot \gamma^2 \cdot \bar{\mathbf{K}})] \right. \\
&\quad \left. + (-2e^{-2A})^2[-2\Gamma_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2] \right], \\
& {}^{(d)}(\Gamma^2 \mathbf{R})^i_j {}^{(d)}(\Gamma \mathbf{R})^j_i = (-4\partial A \bar{\partial} A dz \wedge d\bar{z}) \wedge (-2e^{-2A})^2 [-(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge [(\hat{D}\mathbf{K} \cdot \bar{\mathbf{K}}) + (\hat{D}\bar{\mathbf{K}} \cdot \mathbf{K})]].
\end{aligned} \tag{D.19}$$

Then, by summing all of these terms, after integration, we obtain the following expression:

$$\begin{aligned}
& \int_{\text{Cone}_{\Sigma,n}} \text{tr}(\mathbf{\Gamma} \wedge \mathbf{R} \wedge \mathbf{\Gamma}^2 \wedge \mathbf{R})|_{\partial A \bar{\partial} A} \\
&= (4\pi\epsilon) \int_{\Sigma} \left[-\mathbf{R}_N \wedge [(\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K}) + (\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}})] + \mathbf{\Gamma}_N \wedge [(\bar{\mathbf{K}} \cdot \gamma^2 \cdot \mathbf{K}) + (\mathbf{K} \cdot \gamma^2 \cdot \bar{\mathbf{K}}) - 2(\bar{\mathbf{K}} \cdot \mathbf{K})^2] \right. \\
&\quad + 3(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge [(\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K}) + (\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}})] + (\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge [(\hat{D}\mathbf{K} \cdot \bar{\mathbf{K}}) + (\hat{D}\bar{\mathbf{K}} \cdot \mathbf{K})] \\
&\quad \left. - [(\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{r} \cdot \mathbf{K}) + (\mathbf{K} \cdot \gamma \cdot \mathbf{r} \cdot \bar{\mathbf{K}}) + (\bar{\mathbf{K}} \cdot \gamma^2 \cdot \hat{D}\mathbf{K}) + (\mathbf{K} \cdot \gamma^2 \cdot \hat{D}\bar{\mathbf{K}})] \right]. \quad (\text{D.20})
\end{aligned}$$

(3) ∂A or $\bar{\partial} A$ Term

All the terms of this form contain \mathbf{K} and $\bar{\mathbf{K}}$. Therefore, following the argument in §4.3, we conclude that there is no contribution of this type:

$$\int_{\text{Cone}_{\Sigma,n}} \text{tr}(\mathbf{\Gamma} \wedge \mathbf{R} \wedge \mathbf{\Gamma}^2 \wedge \mathbf{R})|_{\partial A, \bar{\partial} A} = 0. \quad (\text{D.21})$$

Summary for $\text{tr}(\mathbf{\Gamma} \wedge \mathbf{R} \wedge \mathbf{\Gamma}^2 \wedge \mathbf{R})$

By summing up Eqs. (D.17), (D.20) and (D.21), the order- ϵ contribution from $\text{tr}(\mathbf{\Gamma} \wedge \mathbf{R} \wedge \mathbf{\Gamma}^2 \wedge \mathbf{R})$ turns out to be

$$\begin{aligned}
& \left[\int_{\text{Cone}_{\Sigma,n}} \text{tr}(\mathbf{\Gamma} \wedge \mathbf{R} \wedge \mathbf{\Gamma}^2 \wedge \mathbf{R}) \right]_{\epsilon} \\
&= (4\pi\epsilon) \int_{\Sigma} \left[-4\mathbf{\Gamma}_N \wedge \mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) - \mathbf{R}_N \wedge [(\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K}) + (\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}})] \right. \\
&\quad + \mathbf{\Gamma}_N \wedge \left[-2[(\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{K}) + (\mathbf{K} \cdot \mathbf{r} \cdot \bar{\mathbf{K}})] + [(\bar{\mathbf{K}} \cdot \gamma^2 \cdot \mathbf{K}) + (\mathbf{K} \cdot \gamma^2 \cdot \bar{\mathbf{K}})] + 14(\bar{\mathbf{K}} \cdot \mathbf{K})^2 \right. \\
&\quad \left. \left. - [(\hat{D}\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K}) + (\hat{D}\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}}) + (\bar{\mathbf{K}} \cdot \gamma \cdot \hat{D}\mathbf{K}) + (\mathbf{K} \cdot \gamma \cdot \hat{D}\bar{\mathbf{K}})] \right] \right. \\
&\quad + 7(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge [(\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K}) + (\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}})] + 5(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge [(\hat{D}\bar{\mathbf{K}} \cdot \mathbf{K}) + (\hat{D}\mathbf{K} \cdot \bar{\mathbf{K}})] \\
&\quad - (\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \gamma \cdot \mathbf{K}) + (\mathbf{K} \cdot \mathbf{r} \cdot \gamma \cdot \bar{\mathbf{K}}) - 2(\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{r} \cdot \mathbf{K}) + 2(\mathbf{K} \cdot \gamma \cdot \mathbf{r} \cdot \bar{\mathbf{K}}) \\
&\quad \left. \left. - (\bar{\mathbf{K}} \cdot \gamma^2 \cdot \hat{D}\mathbf{K}) + (\mathbf{K} \cdot \gamma^2 \cdot \hat{D}\bar{\mathbf{K}}) \right] \right]. \quad (\text{D.22})
\end{aligned}$$

D.1.4 $\text{tr}(\mathbf{\Gamma}^3 \wedge \mathbf{R}^2)$

We next consider $\text{tr}(\mathbf{\Gamma}^3 \wedge \mathbf{R}^2) = (\mathbf{\Gamma}^2 \mathbf{R})^\mu{}_\nu \wedge (\mathbf{R} \mathbf{\Gamma})^\nu{}_\mu$.

(1) $\partial \bar{\partial} A$ Term

In this case, the followings are potential nontrivial terms:

$$\begin{aligned}
& (d^2)(\Gamma^2 \mathbf{R})^z{}_z{}^{(0)}(\mathbf{R}\Gamma)^z{}_z, & (d^2)(\Gamma^2 \mathbf{R})^{\bar{z}}{}_{\bar{z}}{}^{(0)}(\mathbf{R}\Gamma)^{\bar{z}}{}_{\bar{z}}, \\
& (d^2)(\Gamma^2 \mathbf{R})^i{}_z{}^{(0)}(\mathbf{R}\Gamma)^z{}_i, & (d^2)(\Gamma^2 \mathbf{R})^i{}_{\bar{z}}{}^{(0)}(\mathbf{R}\Gamma)^{\bar{z}}{}_i, \\
& (0)(\Gamma^2 \mathbf{R})^z{}_z{}^{(d^2)}(\mathbf{R}\Gamma)^z{}_z, & (0)(\Gamma^2 \mathbf{R})^{\bar{z}}{}_{\bar{z}}{}^{(d^2)}(\mathbf{R}\Gamma)^{\bar{z}}{}_{\bar{z}}, \\
& (0)(\Gamma^2 \mathbf{R})^j{}_z{}^{(d^2)}(\mathbf{R}\Gamma)^z{}_j, & (0)(\Gamma^2 \mathbf{R})^j{}_{\bar{z}}{}^{(d^2)}(\mathbf{R}\Gamma)^{\bar{z}}{}_j.
\end{aligned} \tag{D.23}$$

We can evaluate these terms one by one as follows:

$$\begin{aligned}
& (d^2)(\Gamma^2 \mathbf{R})^z{}_z{}^{(0)}(\mathbf{R}\Gamma)^z{}_z \\
&= (-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge \left[(-2e^{-2A})\Gamma_N \wedge \mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) \right. \\
&\quad \left. + (-2e^{-2A})^2[\Gamma_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 + (\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\hat{D}\bar{\mathbf{K}} \cdot \mathbf{K})] \right], \\
& (d^2)(\Gamma^2 \mathbf{R})^{\bar{z}}{}_{\bar{z}}{}^{(0)}(\mathbf{R}\Gamma)^{\bar{z}}{}_{\bar{z}} \\
&= (-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge \left[(-2e^{-2A})\Gamma_N \wedge \mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) \right. \\
&\quad \left. + (-2e^{-2A})^2[\Gamma_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 + (\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\hat{D}\mathbf{K} \cdot \bar{\mathbf{K}})] \right], \\
& (d^2)(\Gamma^2 \mathbf{R})^i{}_z{}^{(0)}(\mathbf{R}\Gamma)^z{}_i \\
&= (-2\partial\bar{\partial}Adz \wedge d\bar{z}) \\
&\quad \wedge \left[(-2e^{-2A})[\Gamma_N \wedge \mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) + \mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K}) + \Gamma_N \wedge (\hat{D}\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K}) + (\hat{D}\bar{\mathbf{K}} \cdot \gamma^2 \cdot \mathbf{K})] \right. \\
&\quad \left. + (-2e^{-2A})^2[\Gamma_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 + (\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K})] \right], \\
& (d^2)(\Gamma^2 \mathbf{R})^i{}_{\bar{z}}{}^{(0)}(\mathbf{R}\Gamma)^{\bar{z}}{}_i \\
&= (-2\partial\bar{\partial}Adz \wedge d\bar{z}) \\
&\quad \wedge \left[(-2e^{-2A})[\Gamma_N \wedge \mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) + \mathbf{R}_N \wedge (\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}}) + \Gamma_N \wedge (\hat{D}\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}}) - (\hat{D}\mathbf{K} \cdot \gamma^2 \cdot \bar{\mathbf{K}})] \right. \\
&\quad \left. + (-2e^{-2A})^2[\Gamma_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 + (\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}})] \right],
\end{aligned}$$

$$\begin{aligned}
& {}^{(0)}(\mathbf{\Gamma}^2 \mathbf{R})^z{}_z{}^{(d^2)}(\mathbf{R}\mathbf{\Gamma})^z{}_z \\
&= (-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge \left[(-2e^{-2A})[\mathbf{\Gamma}_N \wedge \mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) + \mathbf{\Gamma}_N \wedge (\bar{\mathbf{K}} \cdot \gamma \cdot \hat{D}\mathbf{K})] + (-2e^{-2A})^2 \mathbf{\Gamma}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 \right], \\
& {}^{(0)}(\mathbf{\Gamma}^2 \mathbf{R})^{\bar{z}}{}_{\bar{z}}{}^{(d^2)}(\mathbf{R}\mathbf{\Gamma})^{\bar{z}}{}_{\bar{z}} \\
&= (-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge \left[(-2e^{-2A})[\mathbf{\Gamma}_N \wedge \mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) + \mathbf{\Gamma}_N \wedge (\mathbf{K} \cdot \gamma \cdot \hat{D}\bar{\mathbf{K}})] + (-2e^{-2A})^2 \mathbf{\Gamma}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 \right], \\
& {}^{(0)}(\mathbf{\Gamma}^2 \mathbf{R})^j{}_z{}^{(d^2)}(\mathbf{R}\mathbf{\Gamma})^z{}_j \\
&= (-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge \left[(-2e^{-2A})[\mathbf{\Gamma}_N \wedge \mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) + \mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K}) + (\bar{\mathbf{K}} \cdot \gamma^2 \cdot \hat{D}\mathbf{K})] \right. \\
&\quad \left. + (-2e^{-2A})^2 [\mathbf{\Gamma}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 + (\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge [(\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K}) + (\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K})]] \right], \\
& {}^{(0)}(\mathbf{\Gamma}^2 \mathbf{R})^j{}_{\bar{z}}{}^{(d^2)}(\mathbf{R}\mathbf{\Gamma})^{\bar{z}}{}_j \\
&= (-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge \left[(-2e^{-2A})[\mathbf{\Gamma}_N \wedge \mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) + \mathbf{R}_N \wedge (\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}}) - (\mathbf{K} \cdot \gamma^2 \cdot \hat{D}\bar{\mathbf{K}})] \right. \\
&\quad \left. + (-2e^{-2A})^2 [\mathbf{\Gamma}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 + (\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge [(\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}}) + (\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}})]] \right].
\end{aligned} \tag{D.24}$$

By summing them and doing the integration, we finally obtain the total of this type of contribution as

$$\begin{aligned}
& \int_{\text{Cone}_{\Sigma,n}} \text{tr}(\mathbf{\Gamma}^3 \wedge \mathbf{R}^2) |_{\partial\bar{\partial}A} \\
&= (4\pi\epsilon) \int_{\Sigma} \left[-8\mathbf{\Gamma}_N \wedge \mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) - 2\mathbf{R}_N \wedge [(\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K}) + (\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}})] \right. \\
&\quad + \mathbf{\Gamma}_N \wedge [16(\bar{\mathbf{K}} \cdot \mathbf{K})^2 - [(\hat{D}\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K}) + (\hat{D}\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}}) + (\bar{\mathbf{K}} \cdot \gamma \cdot \hat{D}\mathbf{K}) + (\mathbf{K} \cdot \gamma \cdot \hat{D}\bar{\mathbf{K}})]] \\
&\quad + 4(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge [(\hat{D}\bar{\mathbf{K}} \cdot \mathbf{K}) + (\hat{D}\mathbf{K} \cdot \bar{\mathbf{K}})] + 4(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge [(\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K}) + (\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}})] \\
&\quad \left. - [(\hat{D}\bar{\mathbf{K}} \cdot \gamma^2 \cdot \mathbf{K}) - (\hat{D}\mathbf{K} \cdot \gamma^2 \cdot \bar{\mathbf{K}}) + (\bar{\mathbf{K}} \cdot \gamma^2 \cdot \hat{D}\mathbf{K}) - (\mathbf{K} \cdot \gamma^2 \cdot \hat{D}\bar{\mathbf{K}})] \right].
\end{aligned} \tag{D.25}$$

(2) $\partial A \bar{\partial} A$ Term

The potential nontrivial terms of this type are as follows:

$$\begin{aligned}
& (dd)(\Gamma^2 \mathbf{R})^i_j{}^{(0)}(\mathbf{R}\Gamma)^j_i, \\
& (d)(\Gamma^2 \mathbf{R})^z{}_z{}^{(d)}(\mathbf{R}\Gamma)^z{}_z, \quad (d)(\Gamma^2 \mathbf{R})^{\bar{z}}{}_{\bar{z}}{}^{(d)}(\mathbf{R}\Gamma)^{\bar{z}}{}_{\bar{z}}, \\
& (d)(\Gamma^2 \mathbf{R})^z{}_j{}^{(d)}(\mathbf{R}\Gamma)^j{}_z, \quad (d)(\Gamma^2 \mathbf{R})^{\bar{z}}{}_j{}^{(d)}(\mathbf{R}\Gamma)^j{}_{\bar{z}}, \\
& (d)(\Gamma^2 \mathbf{R})^i{}_z{}^{(d)}(\mathbf{R}\Gamma)^z{}_i, \quad (d)(\Gamma^2 \mathbf{R})^i{}_{\bar{z}}{}^{(d)}(\mathbf{R}\Gamma)^{\bar{z}}{}_i, \\
& (d)(\Gamma^2 \mathbf{R})^i{}_j{}^{(d)}(\mathbf{R}\Gamma)^j{}_i.
\end{aligned} \tag{D.26}$$

We can compute these quantities directly as

$$\begin{aligned}
& (dd)(\Gamma^2 \mathbf{R})^i_j{}^{(0)}(\mathbf{R}\Gamma)^j_i \\
& = (-4\partial A \bar{\partial} A dz \wedge d\bar{z}) \wedge \left[(-2e^{-2A})[(\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \gamma \cdot \mathbf{K}) - (\mathbf{K} \cdot \mathbf{r} \cdot \gamma \cdot \bar{\mathbf{K}})] \right. \\
& \quad \left. + (-2e^{-2A})^2(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge [(\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) + (\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}}) + (\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K}) + (\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}})] \right], \\
& (d)(\Gamma^2 \mathbf{R})^z{}_z{}^{(d)}(\mathbf{R}\Gamma)^z{}_z \\
& = (-4\partial A \bar{\partial} A dz \wedge d\bar{z}) \wedge (-2e^{-2A})^2 \left[\Gamma_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 - (\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge [(\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) - (\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K})] \right], \\
& (d)(\Gamma^2 \mathbf{R})^{\bar{z}}{}_{\bar{z}}{}^{(d)}(\mathbf{R}\Gamma)^{\bar{z}}{}_{\bar{z}} \\
& = (-4\partial A \bar{\partial} A dz \wedge d\bar{z}) \wedge (-2e^{-2A})^2 \left[\Gamma_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 - (\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge [(\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}}) - (\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}})] \right], \\
& (d)(\Gamma^2 \mathbf{R})^z{}_j{}^{(d)}(\mathbf{R}\Gamma)^j{}_z \\
& = (-4\partial A \bar{\partial} A dz \wedge d\bar{z}) \wedge (-2e^{-2A})^2 [-\Gamma_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 + (\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K})], \\
& (d)(\Gamma^2 \mathbf{R})^{\bar{z}}{}_j{}^{(d)}(\mathbf{R}\Gamma)^j{}_{\bar{z}} \\
& = (-4\partial A \bar{\partial} A dz \wedge d\bar{z}) \wedge (-2e^{-2A})^2 [-\Gamma_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 + (\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}})], \\
& (d)(\Gamma^2 \mathbf{R})^i{}_z{}^{(d)}(\mathbf{R}\Gamma)^z{}_i \\
& = (-4\partial A \bar{\partial} A dz \wedge d\bar{z}) \wedge \left[(-2e^{-2A})[\mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K}) + (\bar{\mathbf{K}} \cdot \gamma^3 \cdot \mathbf{K})] + (-2e^{-2A})^2[2(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K})] \right], \\
& (d)(\Gamma^2 \mathbf{R})^i{}_{\bar{z}}{}^{(d)}(\mathbf{R}\Gamma)^{\bar{z}}{}_i \\
& = (-4\partial A \bar{\partial} A dz \wedge d\bar{z}) \wedge \left[(-2e^{-2A})[\mathbf{R}_N \wedge (\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}}) - (\mathbf{K} \cdot \gamma^3 \cdot \bar{\mathbf{K}})] + (-2e^{-2A})^2[2(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}})] \right], \\
& (d)(\Gamma^2 \mathbf{R})^i{}_j{}^{(d)}(\mathbf{R}\Gamma)^j{}_i \\
& = (-4\partial A \bar{\partial} A dz \wedge d\bar{z}) \wedge (-2e^{-2A})^2 \left[-2\Gamma_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 - (\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge [(\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K}) + (\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}})] \right].
\end{aligned} \tag{D.27}$$

By summing all of them, after integration, we finally obtain the total of this type of contri-

bution as

$$\begin{aligned}
& \int_{\text{Cone}_{\Sigma,n}} \text{tr}(\mathbf{\Gamma}^3 \wedge \mathbf{R}^2)|_{\partial A \bar{\partial} A} \\
&= (4\pi\epsilon) \int_{\Sigma} \left[-\mathbf{R}_N \wedge [(\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K}) + (\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}})] - 2\mathbf{\Gamma}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 \right. \\
&\quad + (\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge [(\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) + (\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}})] + 3(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge [(\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K}) + (\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}})] \\
&\quad \left. - [(\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \gamma \cdot \mathbf{K}) - (\mathbf{K} \cdot \mathbf{r} \cdot \gamma \cdot \bar{\mathbf{K}})] - [(\bar{\mathbf{K}} \cdot \gamma^3 \cdot \mathbf{K}) - (\mathbf{K} \cdot \gamma^3 \cdot \bar{\mathbf{K}})] \right].
\end{aligned} \tag{D.28}$$

(3) ∂A or $\bar{\partial} A$ Term

In a similar way to the previous cases, we can see that all terms of this type contain \mathbf{K} and $\bar{\mathbf{K}}$. Therefore, we conclude that there is no contribution of this type:

$$\int_{\text{Cone}_{\Sigma,n}} \text{tr}(\mathbf{\Gamma}^3 \wedge \mathbf{R}^2)|_{\partial A, \bar{\partial} A} = 0. \tag{D.29}$$

Summary for $\text{tr}(\mathbf{\Gamma}^3 \wedge \mathbf{R}^2)$

By summing up Eqs. (D.25), (D.28) and (D.29), we conclude that the order- ϵ contribution from $\text{tr}(\mathbf{\Gamma}^3 \wedge \mathbf{R}^2)$ is given by

$$\begin{aligned}
& \left[\int_{\text{Cone}_{\Sigma,n}} \text{tr}(\mathbf{\Gamma}^3 \wedge \mathbf{R}^2) \right]_{\epsilon} \\
&= (4\pi\epsilon) \int_{\Sigma} \left[-8\mathbf{\Gamma}_N \wedge \mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) - 3\mathbf{R}_N \wedge [(\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K}) + (\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}})] \right. \\
&\quad + \mathbf{\Gamma}_N \wedge \left[14(\bar{\mathbf{K}} \cdot \mathbf{K})^2 - [(\hat{D}\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K}) + (\hat{D}\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}}) + (\bar{\mathbf{K}} \cdot \gamma \cdot \hat{D}\mathbf{K}) + (\mathbf{K} \cdot \gamma \cdot \hat{D}\bar{\mathbf{K}})] \right] \\
&\quad + 5(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge [(\hat{D}\bar{\mathbf{K}} \cdot \mathbf{K}) + (\hat{D}\mathbf{K} \cdot \bar{\mathbf{K}})] + 7(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge [(\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K}) + (\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}})] \\
&\quad - [(\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \gamma \cdot \mathbf{K}) - (\mathbf{K} \cdot \mathbf{r} \cdot \gamma \cdot \bar{\mathbf{K}})] - [(\bar{\mathbf{K}} \cdot \gamma^3 \cdot \mathbf{K}) - (\mathbf{K} \cdot \gamma^3 \cdot \bar{\mathbf{K}})] \\
&\quad \left. - [(\hat{D}\bar{\mathbf{K}} \cdot \gamma^2 \cdot \mathbf{K}) - (\hat{D}\mathbf{K} \cdot \gamma^2 \cdot \bar{\mathbf{K}}) + (\bar{\mathbf{K}} \cdot \gamma^2 \cdot \hat{D}\mathbf{K}) - (\mathbf{K} \cdot \gamma^2 \cdot \hat{D}\bar{\mathbf{K}})] \right].
\end{aligned} \tag{D.30}$$

D.1.5 $\text{tr}(\mathbf{\Gamma} \wedge \mathbf{R}^3)$

In the end, here we evaluate $\text{tr}(\mathbf{\Gamma} \wedge \mathbf{R}^3) = (\mathbf{\Gamma} \mathbf{R})^\mu{}_\nu \wedge (\mathbf{R}^2)^\nu{}_\mu$ on the regularized cone background.

(1) $\partial \bar{\partial} A$ Term

The followings are potential nontrivial terms of this type:

$$\begin{aligned}
& (d^2)(\Gamma \mathbf{R})^z_z ({}^{(0)}(\mathbf{R}^2)^z_z, & (d^2)(\Gamma \mathbf{R})^{\bar{z}}_{\bar{z}} ({}^{(0)}(\mathbf{R}^2)^{\bar{z}}_{\bar{z}}, \\
& (d^2)(\Gamma \mathbf{R})^i_z ({}^{(0)}(\mathbf{R}^2)^z_i, & (d^2)(\Gamma \mathbf{R})^i_{\bar{z}} ({}^{(0)}(\mathbf{R}^2)^{\bar{z}}_i, \\
& ({}^{(0)}(\Gamma \mathbf{R})^z_z (d^2)(\mathbf{R}^2)^z_z, & ({}^{(0)}(\Gamma \mathbf{R})^{\bar{z}}_{\bar{z}} (d^2)(\mathbf{R}^2)^{\bar{z}}_{\bar{z}}, \\
& ({}^{(0)}(\Gamma \mathbf{R})^j_z (d^2)(\mathbf{R}^2)^z_j, & ({}^{(0)}(\Gamma \mathbf{R})^j_{\bar{z}} (d^2)(\mathbf{R}^2)^{\bar{z}}_j, \\
& ({}^{(0)}(\Gamma \mathbf{R})^z_i (d^2)(\mathbf{R}^2)^i_z, & ({}^{(0)}(\Gamma \mathbf{R})^{\bar{z}}_i (d^2)(\mathbf{R}^2)^i_{\bar{z}},
\end{aligned}$$

each of which is computed as

$$\begin{aligned}
& (d^2)(\Gamma \mathbf{R})^z_z ({}^{(0)}(\mathbf{R}^2)^z_z \\
& = (-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge \left[\Gamma_N \wedge \mathbf{R}_N^2 + (-2e^{-2A})[2\Gamma_N \wedge \mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) + \Gamma_N \wedge (\hat{D}\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K})] \right. \\
& \quad \left. + (-2e^{-2A})^2 \Gamma_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 \right], \\
& (d^2)(\Gamma \mathbf{R})^{\bar{z}}_{\bar{z}} ({}^{(0)}(\mathbf{R}^2)^{\bar{z}}_{\bar{z}} \\
& = (-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge \left[\Gamma_N \wedge \mathbf{R}_N^2 + (-2e^{-2A})[2\Gamma_N \wedge \mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) + \Gamma_N \wedge (\hat{D}\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K})] \right. \\
& \quad \left. + (-2e^{-2A})^2 \Gamma_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 \right], \\
& (d^2)(\Gamma \mathbf{R})^i_z ({}^{(0)}(\mathbf{R}^2)^z_i \\
& = (-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge \left[(-2e^{-2A})[\mathbf{R}_N \wedge (\hat{D}\bar{\mathbf{K}} \cdot \mathbf{K}) + (\hat{D}\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{K})] + (-2e^{-2A})^2[2(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\hat{D}\bar{\mathbf{K}} \cdot \mathbf{K})] \right], \\
& (d^2)(\Gamma \mathbf{R})^i_{\bar{z}} ({}^{(0)}(\mathbf{R}^2)^{\bar{z}}_i \\
& = (-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge \left[(-2e^{-2A})[\mathbf{R}_N \wedge (\hat{D}\mathbf{K} \cdot \bar{\mathbf{K}}) - (\hat{D}\mathbf{K} \cdot \mathbf{r} \cdot \bar{\mathbf{K}})] + (-2e^{-2A})^2[2(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\hat{D}\mathbf{K} \cdot \bar{\mathbf{K}})] \right],
\end{aligned} \tag{D.31}$$

$$\begin{aligned}
& {}^{(0)}(\Gamma \mathbf{R})^z_z (d^2)(\mathbf{R}^2)^z_z \\
&= (-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge \left[2\Gamma_N \wedge \mathbf{R}_N^2 + (-2e^{-2A})[4\Gamma_N \wedge \mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) + 2\mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K})] \right. \\
&\quad \left. + (-2e^{-2A})^2[2\Gamma_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 + 2(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K})] \right], \\
& {}^{(0)}(\Gamma \mathbf{R})^{\bar{z}}_{\bar{z}} (d^2)(\mathbf{R}^2)^{\bar{z}}_{\bar{z}} \\
&= (-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge \left[2\Gamma_N \wedge \mathbf{R}_N^2 + (-2e^{-2A})[4\Gamma_N \wedge \mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) + 2\mathbf{R}_N \wedge (\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}})] \right. \\
&\quad \left. + (-2e^{-2A})^2[2\Gamma_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 + 2(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}})] \right], \\
& {}^{(0)}(\Gamma \mathbf{R})^j_z (d^2)(\mathbf{R}^2)^z_j \\
&= (-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge \left[(-2e^{-2A})[\mathbf{R}_N \wedge (\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}}) + (\hat{D}\bar{\mathbf{K}} \cdot \gamma \cdot \hat{D}\mathbf{K})] + (-2e^{-2A})^2(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\hat{D}\bar{\mathbf{K}} \cdot \mathbf{K}) \right], \\
& {}^{(0)}(\Gamma \mathbf{R})^j_{\bar{z}} (d^2)(\mathbf{R}^2)^{\bar{z}}_j \\
&= (-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge \left[(-2e^{-2A})[\mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) - (\hat{D}\mathbf{K} \cdot \gamma \cdot \hat{D}\bar{\mathbf{K}})] + (-2e^{-2A})^2(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\hat{D}\mathbf{K} \cdot \bar{\mathbf{K}}) \right], \\
& {}^{(0)}(\Gamma \mathbf{R})^z_i (d^2)(\mathbf{R}^2)^i_z \\
&= (-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge \left[(-2e^{-2A})[\Gamma_N \wedge (\hat{D}\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) + (\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \hat{D}\mathbf{K})] + (-2e^{-2A})^2(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) \right], \\
& {}^{(0)}(\Gamma \mathbf{R})^{\bar{z}}_i (d^2)(\mathbf{R}^2)^i_{\bar{z}} \\
&= (-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge \left[(-2e^{-2A})[\Gamma_N \wedge (\hat{D}\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) - (\mathbf{K} \cdot \mathbf{r} \cdot \hat{D}\bar{\mathbf{K}})] + (-2e^{-2A})^2(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}}) \right].
\end{aligned} \tag{D.32}$$

After summing these and doing the integration, we finally obtain the total of this type of

contribution as

$$\begin{aligned}
& \int_{\text{Cone}_{\Sigma,n}} \text{tr}(\mathbf{\Gamma} \wedge \mathbf{R}^3)|_{\partial\bar{\partial}A} \\
&= (4\pi\epsilon) \int_{\Sigma} \left[3\mathbf{\Gamma}_N \wedge \mathbf{R}_N^2 - 12\mathbf{\Gamma}_N \wedge \mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) \right. \\
&\quad - 4\mathbf{R}_N \wedge [(\hat{D}\bar{\mathbf{K}} \cdot \mathbf{K}) + (\hat{D}\mathbf{K} \cdot \bar{\mathbf{K}})] - 4\mathbf{\Gamma}_N \wedge (\hat{D}\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) + 12\mathbf{\Gamma}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 \\
&\quad + 12(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge [(\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) + (\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}})] \\
&\quad - [(\hat{D}\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{K}) - (\hat{D}\mathbf{K} \cdot \mathbf{r} \cdot \bar{\mathbf{K}}) + (\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \hat{D}\mathbf{K}) - (\mathbf{K} \cdot \mathbf{r} \cdot \hat{D}\bar{\mathbf{K}})] \\
&\quad \left. - [(\hat{D}\bar{\mathbf{K}} \cdot \gamma \cdot \hat{D}\mathbf{K}) - (\hat{D}\mathbf{K} \cdot \gamma \cdot \hat{D}\bar{\mathbf{K}})] \right].
\end{aligned} \tag{D.33}$$

(2) $\partial A \bar{\partial} A$ Term

In this case, there are following potential nontrivial possibilities:

$$\begin{aligned}
& {}^{(dd)}(\mathbf{\Gamma R})_j^z {}^{(0)}(\mathbf{R}^2)_z^j, & {}^{(dd)}(\mathbf{\Gamma R})_{\bar{j}}^{\bar{z}} {}^{(0)}(\mathbf{R}^2)_{\bar{z}}^j, \\
& {}^{(0)}(\mathbf{\Gamma R})_z^z {}^{(dd)}(\mathbf{R}^2)_z^z, & {}^{(0)}(\mathbf{\Gamma R})_{\bar{z}}^{\bar{z}} {}^{(dd)}(\mathbf{R}^2)_{\bar{z}}^{\bar{z}}, \\
& {}^{(0)}(\mathbf{\Gamma R})_i^j {}^{(dd)}(\mathbf{R}^2)_j^i, & \\
& {}^{(d)}(\mathbf{\Gamma R})_z^z {}^{(d)}(\mathbf{R}^2)_z^z, & {}^{(d)}(\mathbf{\Gamma R})_{\bar{z}}^{\bar{z}} {}^{(d)}(\mathbf{R}^2)_{\bar{z}}^{\bar{z}}, \\
& {}^{(d)}(\mathbf{\Gamma R})_j^z {}^{(d)}(\mathbf{R}^2)_z^j, & {}^{(d)}(\mathbf{\Gamma R})_{\bar{j}}^{\bar{z}} {}^{(d)}(\mathbf{R}^2)_{\bar{z}}^j, \\
& {}^{(d)}(\mathbf{\Gamma R})_z^i {}^{(d)}(\mathbf{R}^2)_i^z, & {}^{(d)}(\mathbf{\Gamma R})_{\bar{z}}^i {}^{(d)}(\mathbf{R}^2)_{\bar{i}}^{\bar{z}}, \\
& {}^{(d)}(\mathbf{\Gamma R})_i^j {}^{(d)}(\mathbf{R}^2)_j^i. &
\end{aligned}$$

We can compute them as

$$\begin{aligned}
& {}^{(dd)}(\Gamma \mathbf{R})^z_j{}^{(0)}(\mathbf{R}^2)^j_z \\
&= (-4\partial A \bar{\partial} A dz \wedge d\bar{z}) \wedge \left[(-2e^{-2A})[\mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) + (\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \hat{D}\mathbf{K})] + (-2e^{-2A})^2[2(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K})] \right], \\
& {}^{(dd)}(\Gamma \mathbf{R})^{\bar{z}}_j{}^{(0)}(\mathbf{R}^2)^j_{\bar{z}} \\
&= (-4\partial A \bar{\partial} A dz \wedge d\bar{z}) \wedge \left[(-2e^{-2A})[\mathbf{R}_N \wedge (\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}}) - (\mathbf{K} \cdot \mathbf{r} \cdot \hat{D}\bar{\mathbf{K}})] + (-2e^{-2A})^2[2(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}})] \right], \\
& {}^{(0)}(\Gamma \mathbf{R})^z_z{}^{(dd)}(\mathbf{R}^2)^z_z \\
&= (-4\partial A \bar{\partial} A dz \wedge d\bar{z}) \wedge \left[(-2e^{-2A})[-\Gamma_N \wedge \mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})] \right. \\
&\quad \left. + (-2e^{-2A})^2[-\Gamma_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 - (\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K})] \right], \\
& {}^{(0)}(\Gamma \mathbf{R})^{\bar{z}}_{\bar{z}}{}^{(dd)}(\mathbf{R}^2)^{\bar{z}}_{\bar{z}} \\
&= (-4\partial A \bar{\partial} A dz \wedge d\bar{z}) \wedge \left[(-2e^{-2A})[-\Gamma_N \wedge \mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})] \right. \\
&\quad \left. + (-2e^{-2A})^2[-\Gamma_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 - (\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}})] \right], \\
& {}^{(0)}(\Gamma \mathbf{R})^j_i{}^{(dd)}(\mathbf{R}^2)^i_j \\
&= (-4\partial A \bar{\partial} A dz \wedge d\bar{z}) \\
&\quad \wedge \left[(-2e^{-2A})[(\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{r} \cdot \mathbf{K}) - (\mathbf{K} \cdot \gamma \cdot \mathbf{r} \cdot \bar{\mathbf{K}})] \right. \\
&\quad \left. + (-2e^{-2A})^2[(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge [(\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K}) + (\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}})] + (\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge [(\hat{D}\bar{\mathbf{K}} \cdot \mathbf{K}) + (\hat{D}\mathbf{K} \cdot \bar{\mathbf{K}})]] \right], \\
& {}^{(d)}(\Gamma \mathbf{R})^z_z{}^{(d)}(\mathbf{R}^2)^z_z \\
&= (-4\partial A \bar{\partial} A dz \wedge d\bar{z}) \wedge \left[(-2e^{-2A})\mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) + (-2e^{-2A})^2[2(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K})] \right], \\
& {}^{(d)}(\Gamma \mathbf{R})^{\bar{z}}_{\bar{z}}{}^{(d)}(\mathbf{R}^2)^{\bar{z}}_{\bar{z}} \\
&= (-4\partial A \bar{\partial} A dz \wedge d\bar{z}) \wedge \left[(-2e^{-2A})\mathbf{R}_N \wedge (\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}}) + (-2e^{-2A})^2[2(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}})] \right],
\end{aligned}$$

$$\begin{aligned}
& {}^{(d)}(\boldsymbol{\Gamma}\mathbf{R})^z_j {}^{(d)}(\mathbf{R}^2)^j_z \\
&= (-4\partial A\bar{\partial}Adz \wedge d\bar{z}) \wedge \left[(-2e^{-2A})[-\boldsymbol{\Gamma}_N \wedge \mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) - \boldsymbol{\Gamma}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{K})] \right. \\
&\quad \left. + (-2e^{-2A})^2[-2\boldsymbol{\Gamma}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2] \right], \\
& {}^{(d)}(\boldsymbol{\Gamma}\mathbf{R})^{\bar{z}}_j {}^{(d)}(\mathbf{R}^2)^j_{\bar{z}} \\
&= (-4\partial A\bar{\partial}Adz \wedge d\bar{z}) \wedge \left[(-2e^{-2A})[-\boldsymbol{\Gamma}_N \wedge \mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) - \boldsymbol{\Gamma}_N \wedge (\mathbf{K} \cdot \mathbf{r} \cdot \bar{\mathbf{K}})] \right. \\
&\quad \left. + (-2e^{-2A})^2[-2\boldsymbol{\Gamma}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2] \right], \\
& {}^{(d)}(\boldsymbol{\Gamma}\mathbf{R})^i_z {}^{(d)}(\mathbf{R}^2)^z_i \\
&= (-4\partial A\bar{\partial}Adz \wedge d\bar{z}) \wedge \left[(-2e^{-2A})[\mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \boldsymbol{\gamma} \cdot \mathbf{K}) + (\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \boldsymbol{\gamma} \cdot \mathbf{K})] + (-2e^{-2A})^2[2(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\bar{\mathbf{K}} \cdot \boldsymbol{\gamma} \cdot \mathbf{K})] \right], \\
& {}^{(d)}(\boldsymbol{\Gamma}\mathbf{R})^i_{\bar{z}} {}^{(d)}(\mathbf{R}^2)^{\bar{z}}_i \\
&= (-4\partial A\bar{\partial}Adz \wedge d\bar{z}) \wedge \left[(-2e^{-2A})[\mathbf{R}_N \wedge (\mathbf{K} \cdot \boldsymbol{\gamma} \cdot \bar{\mathbf{K}}) - (\mathbf{K} \cdot \mathbf{r} \cdot \boldsymbol{\gamma} \cdot \bar{\mathbf{K}})] + (-2e^{-2A})^2[2(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge (\mathbf{K} \cdot \boldsymbol{\gamma} \cdot \bar{\mathbf{K}})] \right], \\
& {}^{(d)}(\boldsymbol{\Gamma}\mathbf{R})^i_j {}^{(d)}(\mathbf{R}^2)^j_i = (-4\partial A\bar{\partial}Adz \wedge d\bar{z}) \wedge (-2e^{-2A})^2 \left[-(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge [(\hat{D}\mathbf{K} \cdot \bar{\mathbf{K}}) + (\hat{D}\bar{\mathbf{K}} \cdot \mathbf{K})] \right]. \tag{D.34}
\end{aligned}$$

Therefore, by summing all of them and doing integration, we finally obtain

$$\begin{aligned}
& \int_{\text{Cone}_{\Sigma,n}} \text{tr}(\boldsymbol{\Gamma} \wedge \mathbf{R}^3)|_{\partial A\bar{\partial}A} \\
&= (4\pi\epsilon) \int_{\Sigma} \left[4\boldsymbol{\Gamma}_N \wedge \mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) - \mathbf{R}_N \wedge [(\bar{\mathbf{K}} \cdot \boldsymbol{\gamma} \cdot \mathbf{K}) + (\mathbf{K} \cdot \boldsymbol{\gamma} \cdot \bar{\mathbf{K}})] \right. \\
&\quad - 2\mathbf{R}_N \wedge [(\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) + (\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}})] + \boldsymbol{\Gamma}_N \wedge [(\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{K}) + (\mathbf{K} \cdot \mathbf{r} \cdot \bar{\mathbf{K}})] - 6\boldsymbol{\Gamma}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 \\
&\quad + 3(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge [(\bar{\mathbf{K}} \cdot \boldsymbol{\gamma} \cdot \mathbf{K}) + (\mathbf{K} \cdot \boldsymbol{\gamma} \cdot \bar{\mathbf{K}})] + 3(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge [(\hat{D}\mathbf{K} \cdot \bar{\mathbf{K}}) + (\hat{D}\bar{\mathbf{K}} \cdot \mathbf{K})] \\
&\quad - [(\bar{\mathbf{K}} \cdot \boldsymbol{\gamma} \cdot \mathbf{r} \cdot \mathbf{K}) - (\mathbf{K} \cdot \boldsymbol{\gamma} \cdot \mathbf{r} \cdot \bar{\mathbf{K}})] + (\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \boldsymbol{\gamma} \cdot \mathbf{K}) - (\mathbf{K} \cdot \mathbf{r} \cdot \boldsymbol{\gamma} \cdot \bar{\mathbf{K}})] \\
&\quad \left. - [(\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \hat{D}\mathbf{K}) - (\mathbf{K} \cdot \mathbf{r} \cdot \hat{D}\bar{\mathbf{K}})] \right]. \tag{D.35}
\end{aligned}$$

(3) ∂A or $\bar{\partial} A$ Term

In this case, there are nontrivial terms of the form

$$\begin{aligned}
& {}^{(d)}(\boldsymbol{\Gamma}\mathbf{R})^z_z (\mathbf{R})^z_z d\boldsymbol{\Gamma}^z_z = (2\partial Adz) \wedge \mathbf{R}_N^2 \wedge d\boldsymbol{\Gamma}^z_z + \dots = (-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge \mathbf{R}_N^2 \wedge \boldsymbol{\Gamma}_N + \dots, \\
& {}^{(d)}(\boldsymbol{\Gamma}\mathbf{R})^{\bar{z}}_{\bar{z}} (\mathbf{R})^{\bar{z}}_{\bar{z}} d\boldsymbol{\Gamma}^{\bar{z}}_{\bar{z}} = (-2\bar{\partial}Ad\bar{z}) \wedge \mathbf{R}_N^2 \wedge d\boldsymbol{\Gamma}^{\bar{z}}_{\bar{z}} + \dots = (-2\partial\bar{\partial}Adz \wedge d\bar{z}) \wedge \mathbf{R}_N^2 \wedge \boldsymbol{\Gamma}_N + \dots. \tag{D.36}
\end{aligned}$$

Here the second equality corresponds to the integration by part explained around Eq. (2.14). Therefore, we finally obtain the total of this type of contribution as

$$\int_{\text{Cone}_{\Sigma,n}} \text{tr}(\mathbf{\Gamma} \wedge \mathbf{R}^3)|_{\partial A, \bar{\partial} A} = (4\pi\epsilon) \int_{\Sigma} \mathbf{\Gamma}_N \wedge \mathbf{R}_N^2. \quad (\text{D.37})$$

Summary for $\text{tr}(\mathbf{\Gamma} \wedge \mathbf{R}^3)$

By summing up Eqs. (D.33), (D.35) and (D.37), we obtain the order- ϵ contribution from $\text{tr}(\mathbf{\Gamma} \wedge \mathbf{R}^3)$ as

$$\begin{aligned} & \left[\int_{\text{Cone}_{\Sigma,n}} \text{tr}(\mathbf{\Gamma} \wedge \mathbf{R}^3) \right]_{\epsilon} \\ &= (4\pi\epsilon) \int_{\Sigma} \left[4\mathbf{\Gamma}_N \wedge \mathbf{R}_N^2 - 8\mathbf{\Gamma}_N \wedge \mathbf{R}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K}) - \mathbf{R}_N \wedge [(\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K}) + (\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}})] \right. \\ & \quad - 6\mathbf{R}_N \wedge [(\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) + (\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}})] - 4\mathbf{\Gamma}_N \wedge (\hat{D}\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) \\ & \quad + \mathbf{\Gamma}_N \wedge [(\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{K}) + (\mathbf{K} \cdot \mathbf{r} \cdot \bar{\mathbf{K}})] + 6\mathbf{\Gamma}_N \wedge (\bar{\mathbf{K}} \cdot \mathbf{K})^2 \\ & \quad + 3(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge [(\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K}) + (\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}})] + 15(\bar{\mathbf{K}} \cdot \mathbf{K}) \wedge [(\hat{D}\mathbf{K} \cdot \bar{\mathbf{K}}) + (\hat{D}\bar{\mathbf{K}} \cdot \mathbf{K})] \\ & \quad - [(\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{r} \cdot \mathbf{K}) - (\mathbf{K} \cdot \gamma \cdot \mathbf{r} \cdot \bar{\mathbf{K}}) + (\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \gamma \cdot \mathbf{K}) - (\mathbf{K} \cdot \mathbf{r} \cdot \gamma \cdot \bar{\mathbf{K}})] \\ & \quad - [(\hat{D}\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{K}) - (\hat{D}\mathbf{K} \cdot \mathbf{r} \cdot \bar{\mathbf{K}}) + 2(\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \hat{D}\mathbf{K}) - 2(\mathbf{K} \cdot \mathbf{r} \cdot \hat{D}\bar{\mathbf{K}})] \\ & \quad \left. - [(\hat{D}\bar{\mathbf{K}} \cdot \gamma \cdot \hat{D}\mathbf{K}) - (\hat{D}\mathbf{K} \cdot \gamma \cdot \hat{D}\bar{\mathbf{K}})] \right]. \end{aligned} \quad (\text{D.38})$$

D.2 Final Result

By substituting Eqs. (D.6), (D.14), (D.22), (D.30), (D.38) into Eq. (D.1), we obtain the order- ϵ contribution to $\mathbf{I}_{\text{CS}}^{7d, \text{single}}$ on the regularized cone background as

$$\begin{aligned} & \left[\int_{\text{Cone}_{\Sigma,n}} \mathbf{I}_{\text{CS}}^{7d, \text{single}} \right]_{\epsilon} \\ &= (16\pi\epsilon) \int_{\Sigma} \left[\mathbf{\Gamma}_N \wedge \mathbf{R}_N^2 + [-2\mathbf{R}_N + 3(\bar{\mathbf{K}} \cdot \mathbf{K})] \wedge [(\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}}) + (\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K})] \right. \\ & \quad + (\hat{D}\mathbf{K} \cdot \mathbf{r} \cdot \bar{\mathbf{K}}) - (\hat{D}\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{K}) \\ & \quad + d \left[\frac{1}{2} \mathbf{\Gamma}_N \wedge [(\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K}) + (\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}})] - \frac{3}{20} \mathbf{\Gamma}_N \wedge [(\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K}) + (\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}})] \right. \\ & \quad \left. \left. + \frac{1}{4} [(\hat{D}\bar{\mathbf{K}} \cdot \gamma \cdot \mathbf{K}) - (\hat{D}\mathbf{K} \cdot \gamma \cdot \bar{\mathbf{K}})] - \frac{3}{20} [(\bar{\mathbf{K}} \cdot \gamma^2 \cdot \mathbf{K}) + (\mathbf{K} \cdot \gamma^2 \cdot \bar{\mathbf{K}})] \right] \right]. \end{aligned} \quad (\text{D.39})$$

Here we have used the fact that $D\gamma^i_j = \mathbf{r}^i_j + (\gamma^2)^i_j$. Therefore, by neglecting the total derivative term, we finally obtain the holographic entanglement entropy formula for the 7d

single-trace Chern-Simons term as follows:

$$S_{\text{ent,CS}}^{7d, \text{single}} = (16\pi) \int_{\Sigma} \left[\Gamma_N \wedge \mathbf{R}_N^2 + [-2\mathbf{R}_N + 3(\bar{\mathbf{K}} \cdot \mathbf{K})] \wedge [(\mathbf{K} \cdot \hat{D}\bar{\mathbf{K}}) + (\bar{\mathbf{K}} \cdot \hat{D}\mathbf{K})] \right. \\ \left. + (\hat{D}\mathbf{K} \cdot \mathbf{r} \cdot \bar{\mathbf{K}}) - (\hat{D}\bar{\mathbf{K}} \cdot \mathbf{r} \cdot \mathbf{K}) \right]. \quad (\text{D.40})$$

This result is consistent with the result obtained from the anomaly polynomial in §3.

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